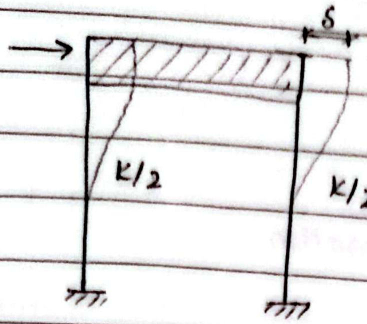
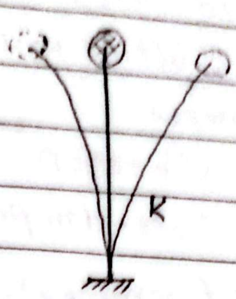
Chapter-2

Single Degree of Freedom (SDOF) Systems:-

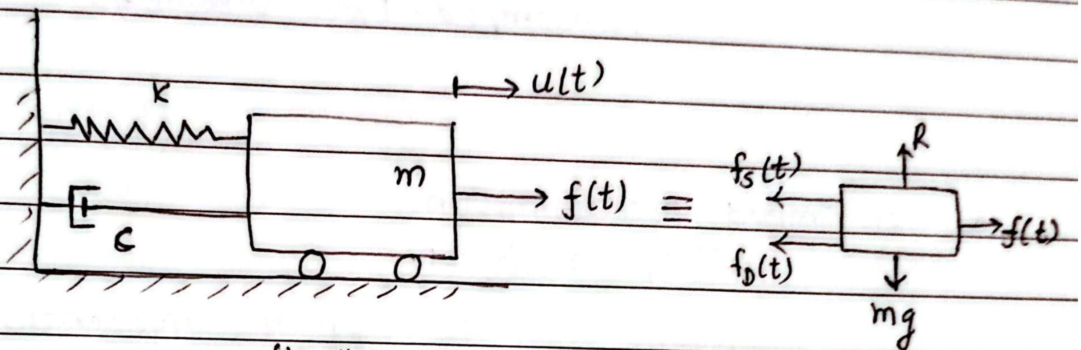
2.1 Equation of Motion & Natural Frequency



i)



(ii)



fig(iii)

$f_s(t) \rightarrow$ spring force
$f_D(t) \rightarrow$ Damping force.
$f_I(t)$ = Inertia force
c = Damping coefficien
k = spring/elastic coefficien

→ For dynamic equilibrium:-

$$\Sigma F = 0$$

For fig.(iii)

$$f(t) - f_s(t) - f_D(t) - f_I(t) = 0 \quad -(\mathrm{I})$$

where,

$f_s(t)$ = Spring/elastic force
$f_D(t)$ = Damping force
$f_I(t)$ = Inertia force
$f(t)$ = Applied force

} at any time 't'

$$f_s(t) = k \cdot u(t)$$

$$f_D(t) = c \cdot \dot{u}(t)$$

$$f_I(t) = m \cdot \ddot{u}(t)$$

Damped vibration (if not undamped vibration,

or, $m\ddot{u}(t) + c.\dot{u}(t) + k.u(t) = f(t)$ — (II)
Forced vibration

If applied force $f(t) = 0$,
$m\ddot{u}(t) + c\dot{u}(t) + k.u(t) = 0$ — (3)
damped
Free vibration
i.e. Free-damped vibration

If c (damping) $= 0$,
$m\ddot{u}(t) + k u(t) = 0$
[Undamped-Free vibration].

* Undamped-free vibration
$m\ddot{u}(t) + k u(t) = 0$
Second-order homogenous differential equation,
Assuming trial solution,
$u = A\cos\omega t$
$\therefore \dot{u} = -\omega A\sin\omega t$ $[\dot{u} = \dot{u}(t)]$.
$\ddot{u} = -\omega^2 A\cos\omega t$

$\therefore m(-\omega^2 A\cos\omega t) + k(A\cos\omega t) = 0$
or, $(-m\omega^2 + k)A\cos\omega t = 0$
As, $A\cos\omega t \neq 0$
Thus,
$-m\omega^2 + k = 0$
or, $\omega^2 = \frac{k}{m}$

or, $\omega = \sqrt{\frac{k}{m}}$ (Taking +ve square root only)
↳ known as natural circular frequency or simply natural frequency

Now, if $W = mg$
if $u_{static} = u_{st} = \frac{W}{k}$ (static displacement due to weight)
Then,
$\omega = \sqrt{\frac{g}{u_{st}}}$ $\left\{\because \sqrt{\frac{mg}{\frac{W/u_{st}}{m}}}\right\}$.

[2.3] Undamped-free vibration response:

$m.\ddot{u}(t) + k u(t) = 0$ — (i)

Solutions:-
$u(t) = e^{st}$ where s being unknown constant.

$\Rightarrow \dot{u}(t) = s e^{st}$
$\Rightarrow \ddot{u}(t) = s^2 e^{st}$
substituting in equation (i)

or, $m s^2 e^{st} + k e^{st} = 0$
or, $(m s^2 + k) e^{st} = 0$
For real value power of e, $e^{st} \neq 0$

or, $ms^2 + k = 0$
or, $s^2 = -\frac{k}{m} = -\omega^2$
or, $s^2 = i^2\omega^2$
or, $[s = \pm i\omega]$
or, $\left.\begin{array}{l} s_1 = i\omega \\ s_2 = -i\omega \end{array}\right\}$.

The general solution of equation (i) is

$$u(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

$$u(t) = A_1 e^{i\omega t} + A_2 e^{-i\omega t} \quad \text{— (ii)}$$

A_1 & A_2 constants to be known,

using,

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \Rightarrow e^{ix} + e^{-ix} = 2\cos x \quad \text{— (3)}$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \Rightarrow e^{ix} - e^{-ix} = 2i\sin x \quad \text{— (4)}$$

Thus,

$$\left.\begin{aligned} e^{ix} &= \cos x + i\sin x \\ e^{-ix} &= \cos x - i\sin x \end{aligned}\right\}.$$

Thus, substituting,

$$u(t) = A_1[\cos\omega t + i\sin\omega t] + A_2[\cos\omega t - i\sin\omega t]$$

or $u(t) = (A_1 + A_2)\cos\omega t + (A_1 - A_2)\, i\sin\omega t$

$$\therefore\ u(t) = A\cos\omega t + Bi\sin\omega t \quad \text{— (5)}$$

where, $A = A_1 + A_2$ (say)

$B = A_1 - A_2$

and,

$$\omega = \sqrt{\frac{k}{m}} \rightarrow \text{natural circular frequency (rad/sec)}$$

$$T = \frac{2\pi}{\omega} \rightarrow \text{Time period}$$

$$\& \quad \frac{1}{T} = f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \rightarrow \text{Natural frequency (Hz)}$$

Now,

$$\dot{u}(t) = -\omega A\sin\omega t + \omega Bi\cos\omega t$$

Initial condition,

at, $t = 0$; $u(t) = u(0)$

& $\dot{u}(t) = \dot{u}(0)$

$$\therefore\ \dot{u}(0) = A\cancel{\sin(0)}^{\,0} + \omega Bi\cos(0)$$

$$\Rightarrow B = \frac{\dot{u}(0)}{\omega i}$$

$$u(0) = A\cos(0) + Bi\cancel{\sin(0)}^{\,0}$$

$$\Rightarrow A = u(0)$$

Hence, equation (5) becomes,

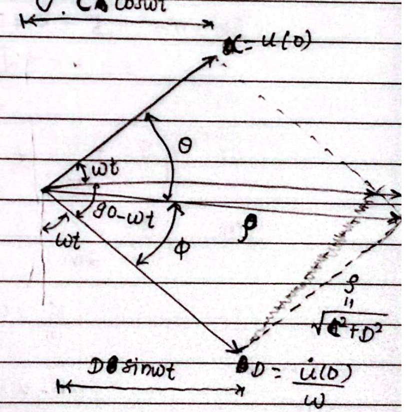
$$\boxed{u(t) = u(0)\cos\omega t + \frac{\dot{u}(0)}{\omega}\sin\omega t} \quad \text{— (*)}$$

Basic equation for undamped-free vibration

- Amplitude of motion is given by,

$$\rho = \sqrt{C^2 + D^2}$$

$$\rho = \sqrt{u(0)^2 + \left[\frac{\dot{u}(0)}{\omega}\right]^2}$$



Direction,

$$\tan\theta = \frac{\dot{u}(0)}{\omega \cdot u(0)}$$

&

$$\tan\phi = \frac{\omega u(0)}{\dot{u}(0)}$$

∴ Response (solution) can also be written as

$$u(t) = \rho \cos(\omega t - \theta)$$
or $$u(t) = \rho \sin(\omega t + \phi)$$

2.5 Damped Free vibration

$$m\ddot{u}(t) + c\dot{u}(t) + ku(t) = 0 \quad \text{---- (12)}$$

We have, General solution as:-

$$u(t) = e^{st}$$
$$\dot{u}(t) = se^{st}$$
$$\ddot{u}(t) = s^2 e^{st}$$

On substituting,

$$ms^2e^{st} + cse^{st} + ke^{st} = 0$$

or, $(ms^2 + cs + k)e^{st} = 0$

As, $e^{st} \neq 0$

So, $ms^2 + cs + k = 0$

$$\therefore s = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

or, $$s = \frac{-c}{2m} \pm \frac{\sqrt{c^2 - 4mk}}{2m}$$

or, $$s = \frac{-c}{2m} \pm \sqrt{\frac{c^2}{4m^2} - \frac{k}{m}} \quad \text{--- (13)}$$

∵ $\frac{k}{m} = \omega^2$, where ω being undamped free-vibration natural frequency.

$$\therefore s = \frac{-c}{2m} \pm \sqrt{\frac{c^2}{4m^2} - \omega^2} \quad \text{--- (14)}$$

→ 3 types of motion possible

$$\left(\frac{c}{2m}\right)^2 > \omega^2 \longrightarrow \text{case I}$$

$$\left(\frac{c}{2m}\right)^2 < \omega^2 \longrightarrow \text{case II}$$

$$\left(\frac{c}{2m}\right)^2 = \omega^2 \longrightarrow \text{case III}$$

- Case III

When, $\left(\frac{c}{2m}\right)^2 = \omega^2$

or, $\frac{c}{2m} = \omega$

or, $[c = 2m \cdot \omega]$ → critical damping coefficient/value.

Thus,

$$[c = c_{cr} = 2m \cdot \omega]$$

$$\left[c_{cr} = 2m\sqrt{\frac{k}{m}} = 2\sqrt{mk} = 2k\sqrt{\frac{m}{k}} = \frac{2k}{\omega}\right] \text{--- (15)}$$

∴ Solution becomes:

$$s = \frac{-c}{2m} = -\omega$$ } Oscillation ceases imediately

* Damping ratio (ξ)

$$\xi = \frac{c}{c_{cr}} = \frac{c}{2m\omega} = \frac{c\omega}{2k}$$

(c: damping; c_{cr}: critical damping)

property of system that depends on:

↳ mass
↳ stiffness
↳ natural frequency
↳ damping

(i) If $c = c_{cr}$ or, $\xi = 1$

↳ the system returns to equilibrium position without oscillating (critically damped system)

(ii) If $c > c_{cr}$ or, $\xi > 1$

↳ again system does not oscillate and return to equilibrium position but at a slower rate than critically damped system [over-damped system]

(iii) If $c < c_{cr}$ or, $\xi < 1$

↳ the system oscillates about its equilibrium position with progressively decreasing amplitude [Under-damped System].

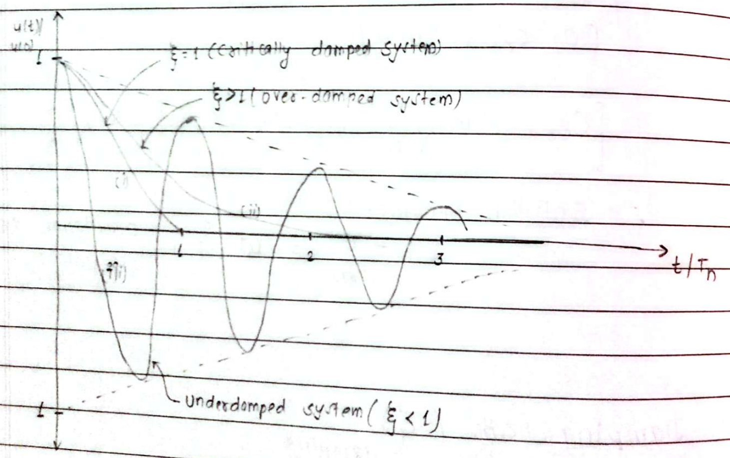


Fig: Showing oscillation of underdamped, overdamped and critically damped systems.

(critically damped condition)

→ $m\ddot{u}(t) + c\dot{u}(t) + k u(t) = 0$

One particular solution,

$u(t) = e^{st}$

Another particular solution,

$u(t) = t e^{st}$

∴ The complete solution is

$u(t) = A_1 e^{st} + A_2 t e^{st}$ ---(16)

or, $u(t) = (A_1 + tA_2) e^{st}$

• For critically damped condition (case III),

$u(t) = (A_1 + tA_2) e^{-\omega t}$

Differentiating,

$\dot{u}(t) = (0 + A_2) e^{-\omega t} + (A_1 + tA_2)(-\omega) e^{-\omega t}$

or $\dot{u}(t) = \{-\omega A_1 + A_2(1 - t\omega)\} e^{-\omega t}$

At initial condition,

at $t=0$; $u(t) = u(0)$ & $\dot{u}(t) = \dot{u}(0)$

$u(0) = (A_1 + 0 \cdot A_2) e^{-\omega \cdot 0} = A_1$

$\dot{u}(0) = (-\omega A_1 + A_2)$

$\Rightarrow A_2 = \dot{u}(0) + \omega u(0)$

Substituting in equation (16)

$u(t) = [u(0)(1 + \omega t) + \dot{u}(0) t] e^{-\omega t}$ —(17)

case I: Over damped system

when $\left(\frac{c}{2m}\right)^2 > \omega^2$

i.e. $\xi = \frac{c}{c_{cr}} > 1$

or $c > c_{cr}$

or $c > 2m$...

or, $\xi = \frac{c}{c_{cr}} = \frac{c}{2m\omega}$

Thus, In equation (14)

$$s = -\xi\omega \pm \sqrt{\xi^2\omega^2 - \omega^2} \qquad \left\{\frac{c}{2m} = \xi m\right\}$$

$$\left\{ s = -\xi\omega \pm \omega\sqrt{\xi^2 - 1} \right.$$

or,

$$s_1 = -\xi\omega + \omega\sqrt{\xi^2-1}$$

$$\& \; s_2 = -\xi\omega - \omega\sqrt{\xi^2-1}$$

or, $s = -\xi\omega \pm \sqrt{-\omega^2(1-\xi^2)}$

or, $s = -\xi\omega \pm i\sqrt{\omega_D^2}$

where,

ω_D = Damped frequency

ω_D^2

$$\rightarrow s = -\xi\omega \pm \omega_D \Big]$$

where,

$$\omega_D^2 = \omega^2(\xi^2 - 1)$$

or, $\omega_D = \omega\sqrt{\xi^2-1}$

$\omega_D \rightarrow$ damped circular frequency

$\therefore$ Hence, final solution in this case:

$$u(t) = e^{-\xi\omega t}(A \sinh\omega t + B\cosh\omega t)$$

• Case II : Under damped system

when, $\left(\frac{c}{2m}\right)^2 < \omega^2$

or, $c < 2m\omega$ or, $c < c_{cr}$

$$\xi = \frac{c}{c_{cr}} = \frac{c}{2m\omega} < 1$$

$$\therefore s = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \omega^2}$$

$$= -\xi\omega \pm \sqrt{\xi^2\omega^2 - \omega^2} \quad -ve$$

$$= -\xi\omega \pm \sqrt{-\omega^2(1-\xi^2)}$$

$$\therefore s = -\xi\omega \pm i\omega_D \Big\}$$

where,

$$\omega_D^2 = \omega^2(1-\xi^2)$$

ω_D = damped circular frequency

The complete solution for under damped free vibration is:

$$u(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

or, $u(t) = A_1 e^{(-\xi\omega + i\omega_D)t} + A_2 e^{(-\xi\omega - i\omega_D)t}$

or, $u(t) = e^{-\xi\omega t}\left[A_1 e^{i\omega_D t} + A_2 e^{-i\omega_D t}\right\}$ =

In trigonometric function,

$$u(t) = e^{-\xi\omega t}\left[A\cos\omega_D t + B\sin\omega_D t\right]$$

After applying Boundary condition,

$$u(t) = e^{-\xi\omega t}\left[u(0)\cos\omega_D t + \left(\frac{\dot{u}(0) + \xi\omega u(0)}{\omega_D}\right)\sin\omega_D t\right]$$

where, $\omega_D = \omega\sqrt{1-\xi^2}$

In terms of amplitude

$$u(t) = \rho e^{-\xi \omega t} \cos(\omega_D t - \theta)$$

where,

$$\rho = \left[\{u(0)\}^2 + \left\{ \frac{\dot{u}(0) + \xi \omega \dot{u}(0)}{\omega_D} \right\}^2 \right]^{1/2}$$

and

$$\theta = \tan^{-1} \left[\frac{\dot{u}(0) + \xi \omega \dot{u}(0)}{\omega_D \cdot u(0)} \right].$$

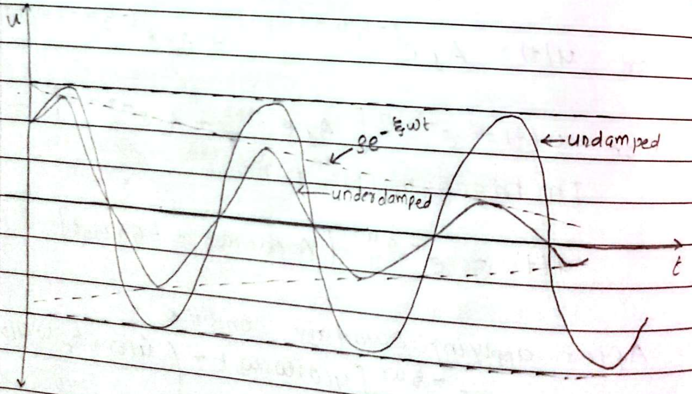
Damped time period (T_D):

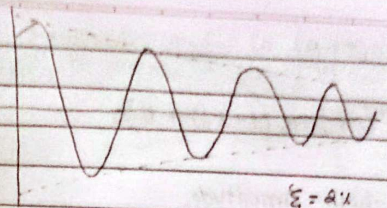
$$T_D = \frac{2\pi}{\omega_D} \therefore \frac{2\pi}{\omega\sqrt{1-\xi^2}} = \frac{T}{\sqrt{1-\xi^2}}$$

Damped circular frequency (ω_D):

$$\omega_D = \omega\sqrt{1-\xi^2}$$

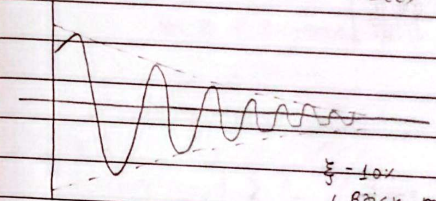
$$\left.\begin{aligned} &\frac{T}{T_D} = \sqrt{1-\xi^2} = \frac{\omega_D}{\omega} \\ &\text{or, } \boxed{\frac{T}{T_D} = \frac{\omega_D}{\omega}} \end{aligned}\right\} \text{Relation b/w underdamped \& undamped case}$$



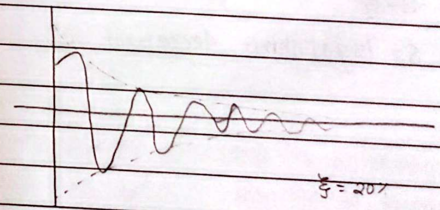


$\xi = 2\%$
(Steel)

$\xi = 5\%$
(RCC)



$\xi = 10\%$
(Brick masonry & stone masonry)



$\xi = 20\%$
(wood)

• $u(t) = \rho e^{-\xi \omega t} \cos(\omega_D t + \theta)$ → same cosine curve but seen at T_D time period difference

Put $t = t + T_D$

$u(t+T_D) = \rho e^{-\xi\omega(t+T_D)} \cos\{\omega_D (t+t_D) + \theta\}$

As, we are talking about amplitude,

$$\frac{u(t)}{u(t+T_D)} = \frac{e^{-\xi\omega t}}{e^{-\xi\omega(t+T_D)}}$$

$$\Rightarrow \frac{u(t)}{u(t+T_D)} = e^{+\xi\omega T_D}$$

$$\text{or, } \frac{u(t)}{u(t+T_D)} = e^{\left[\frac{2\pi\omega\xi}{\omega_D}\right]} = e^{\frac{2\pi\cdot\xi}{\sqrt{1-\xi^2}}}$$

$$\text{i.e. } \frac{u_n}{u_{n+1}} = e^{\left[\frac{2\pi\xi}{\sqrt{1-\xi^2}}\right]}$$

Also,

In logarithmic,

$$\left[\ln\left[\frac{u_n}{u_{n+1}}\right] = \frac{2\pi\xi}{\sqrt{1-\xi^2}} = \delta\right]$$

where, δ = logarithmic decrement

Now

$\frac{u_1}{u_5} = ?$ $u_1 \to$ Amplitude in 1^{st} cycle

$u_5 \to$ Amplitude in 5^{th} cycle

$$\frac{u_1}{u_5} = \frac{u_1}{u_2}\cdot\frac{u_2}{u_3}\cdot\frac{u_3}{u_4}\cdot\frac{u_4}{u_5}$$

$$= e^{\delta}\cdot e^{\delta}\cdot e^{\delta}\cdot e^{\delta}$$

$$= e^{4\delta}$$ 4 → difference in no. of cycle

$$\boxed{\cdot}\ \frac{u_n}{u_{(n+m)}} = e^{m\delta}$$

$$\therefore m\delta = \ln\left[\frac{u_n}{u_{(n+m)}}\right]$$

$$\text{or, } m = \frac{1}{\delta}\ln\left[\frac{u_n}{u_{(n+m)}}\right]$$

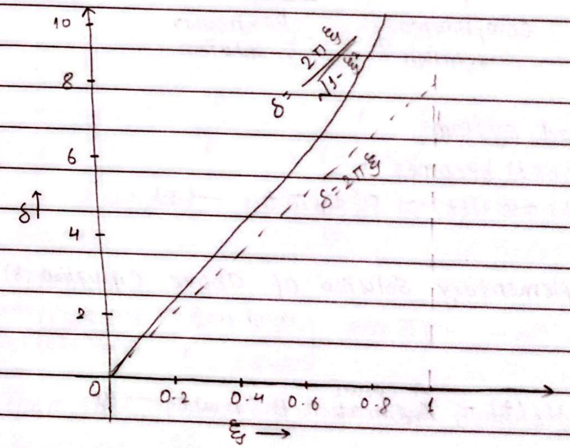
Q. After how many cycles displacement will decrease by 50%?

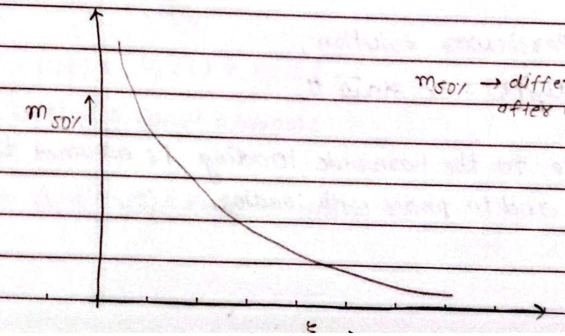
→ $u_{(n+m)} = 0.5\, u_n$

$$\therefore m = \frac{\sqrt{1-\xi^2}}{2\pi\xi}\cdot\ln\left\{\frac{1}{0.5}\right\}$$

for $\xi = 5\% = 0.05$ (concrete)

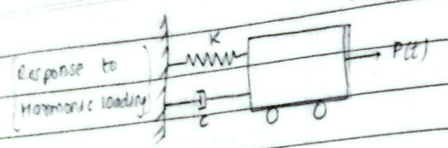
$m \approx 2.2$ cycles





$m_{50\%}$ → difference in no. of cycles after which displacement will reduce by 50%

SDOF :- Forced vibration:



$P(t) = P_0 \sin \bar{\omega} t$

$\bar{\omega}$ = forcing frequency or exciting frequency

Equation of motion in case of harmonic loading.

$m\ddot{u}(t) + c\dot{u}(t) + k u(t) = P_0 \sin \bar{\omega} t \quad —(1)$

The general solution is:

$u(t) = u_c(t) + u_p(t) \quad —(2)$

Complimentary solution — Particular solution

For undamped system,

equation (1) becomes

$m\ddot{u}(t) + k\,u(t) = P_0 \sin \bar{\omega} t \quad —(3)$

∴ Complementary solution of above equation (3) will be

{right side को zero i.e free vibration}

$u_c(t) = A\cos\omega t + B \sin \omega t \quad —(4)$

No $\bar{\omega}$??

And Particular solution,

$u_p(t) = C \sin \bar{\omega} t$

The response to the harmonic loading is assumed to be harmonic and in phase with loading. {$\sin \bar{\omega} t$}

$\dot{u}_p(t) = \bar{\omega} C \cos(\bar{\omega} t)$

$\ddot{u}_p(t) = -C\bar{\omega}^2 \sin \bar{\omega} t$

So, equation (3) becomes

$-mC\bar{\omega}^2 \sin \bar{\omega} t + kC \sin \bar{\omega} t = P_0 \sin \bar{\omega} t$

As, $\sin \bar{\omega} t \neq 0$; we have

$-mC\bar{\omega}^2 + kC = P_0$

or, $(-m\bar{\omega}^2 + k)C = P_0$

$\left\{ \because \frac{k}{m} = \omega^2 \right\}$

$\therefore \left(-\frac{\bar{\omega}^2}{\omega^2} + 1 \right) C = \frac{P_0}{k}$

So, $C = \frac{P_0}{k} \frac{1}{\left(-\frac{\bar{\omega}^2}{\omega^2} + 1 \right)}$

$\therefore C = \frac{P_0}{k} \cdot \left(\frac{1}{1-\beta^2} \right)$

where, $\beta = \frac{\bar{\omega}}{\omega}$ is frequency ratio $= \left\{ \frac{\text{Exciting frequency}}{\text{Natural frequency of system}} \right\}$

Hence,

$u_p(t) = \frac{P_0}{k} \cdot \frac{1}{(1-\beta^2)} \sin \bar{\omega} t \quad —(5)$

Then, the general solution of harmonic excitation of SDOF is given by,

$u(t) = u_c(t) + u_p(t)$

$\therefore \boxed{u(t) = A\cos\omega t + B\sin\omega t + \frac{P_0}{k} \cdot \frac{1}{(1-\beta^2)} \sin \bar{\omega} t} \quad —(6)$

$\dot{u}(t) = -\omega A \sin\omega t + \omega B \cos\omega t + \frac{P_0}{k} \cdot \frac{1}{(1-\beta^2)} \bar{\omega} \cos(\bar{\omega} t)$

At, $t=0$

$u(0) = A$

$\Rightarrow A = u(0)$

At, $t=0$, $u(t) = u(0)$ & $\dot{u}(t) = \dot{u}(0)$

or $\dot{u}(0) = \omega B + \frac{P_0}{K} \cdot \frac{1}{(1-\beta^2)} \bar{\omega}$

or, $B = \left(\frac{\dot{u}(0)}{\omega} - \frac{P_0}{K} \cdot \frac{\beta}{(1-\beta^2)} \right)$ $\left(\frac{\bar{\omega}}{\omega}\right)$

Substituting in equation ⑥,

Solution with initial condition

$$u(t) = \underbrace{u(0)\cos\omega t + \left[\frac{\dot{u}(0)}{\omega} - \frac{P_0}{K} \cdot \frac{\beta}{(1-\beta^2)} \right] \sin\omega t}_{u_c(t)} + \underbrace{\frac{P_0}{K} \cdot \frac{1}{(1-\beta^2)} \sin\bar{\omega}t}_{u_p(t)} \quad \text{—(7)}$$

$u_c(t)$ (complementary solution)

Free vibration/Transient response

$u_p(t)$

(particular solution)

(Forced vibration)

→ steady-state response

(do not depend on initial condition but depend on loading or exciting condition).

[constant response]

⇒ If the system starts from rest at origin

i.e. $u(0) = \dot{u}(0) = 0$

i.e. $A = 0$

$B = -\frac{P_0}{K} \cdot \frac{\beta}{(1-\beta^2)}$

Hence, the response become:

$$u(t) = \frac{P_0}{K} \cdot \frac{1}{(1-\beta^2)} \{ \sin\bar{\omega}t - \beta \sin\omega t \} \quad \text{— ⑧}$$

$\frac{P_0}{k} = u_{st}$ (static displacement)

$\frac{1}{(1-\beta^2)}$ = magnification factor (D)

(dynamic amplification factor)

$\beta = 1$ → natural frequency matches exciting frequency

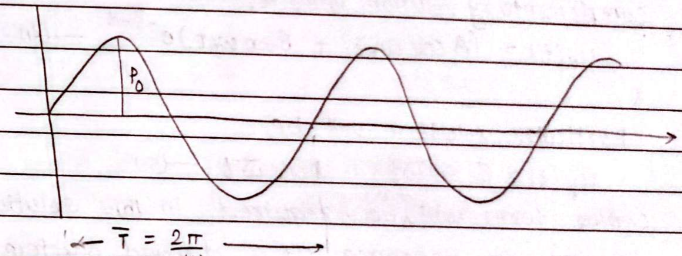
leading to $u(t) \rightarrow \infty$ leading to resonance & failure

Response ratio

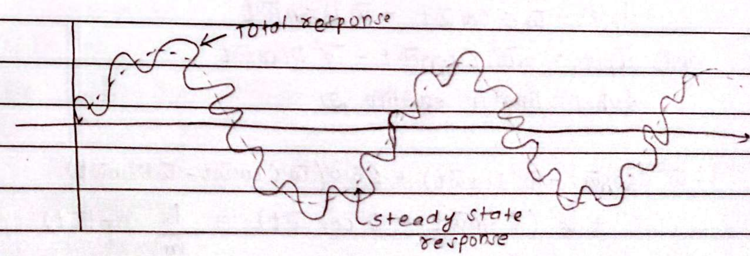
$$R(t) = \frac{u(t)}{u_{st}} = \frac{u(t)}{P_0/K} \quad \text{— ⑨}$$

$$\therefore R(t) = \frac{1}{(1-\beta^2)} \{ \sin\bar{\omega}t - \beta \sin\omega t \} \quad \text{—(10)}$$

starts from rest condition



Harmonic loading



Harmonic Response of Damped System (Forced Vibration)

$$m\ddot{u}(t) + c\dot{u}(t) + k\,u(t) = P_0 \sin\bar{\omega}t \quad \text{—(1)}$$

$$\xi = \frac{c}{c_c} = \frac{c}{2m\omega}$$

$$\Rightarrow \frac{c}{m} = 2\xi\omega, \quad \frac{k}{m} = \omega^2$$

So,

$$\ddot{u}(t) + 2\xi\omega\,\dot{u}(t) + \omega^2 u(t) = \frac{P_0}{m}\sin(\bar{\omega}t) \quad \text{—(2)}$$

General solution will be

$$u(t) = u_c(t) + u_p(t) \quad \text{—(3)}$$

complementary solution, particular solution

Complementary solution will be,

$$u_c(t) = (A\cos\omega_D t + B\sin\omega_D t)e^{-\xi\omega t} \quad \text{—(4)}$$

Particular solution will be,

$$u_p(t) = C\sin\bar{\omega}t + D\cos\bar{\omega}t \quad \text{—(5)}$$

Cosine term will also be required in this solution because the response of a damped system is not in the phase with the loading:-

$$\dot{u}_p(t) = \bar{\omega}C\cos\bar{\omega}t - \bar{\omega}D\sin\bar{\omega}t$$

and $\ddot{u}_p(t) = -\bar{\omega}^2 C\sin\bar{\omega}t - \bar{\omega}^2 D\cos\bar{\omega}t$

Substituting in equation (2)

$$(-\bar{\omega}^2 C\sin\bar{\omega}t - \bar{\omega}^2 D\cos\bar{\omega}t) + 2\xi\omega(\bar{\omega}C\cos\bar{\omega}t - \bar{\omega}D\sin\bar{\omega}t) + \omega^2(C\sin\bar{\omega}t + D\cos\bar{\omega}t) = \frac{P_0}{m}\sin(\bar{\omega}t)$$

or, $$\{(\omega^2 - \bar{\omega}^2)C - 2\xi\omega\bar{\omega}D\}\sin\bar{\omega}t + \{2\xi\omega\bar{\omega}C + (\omega^2 - \bar{\omega}^2)D\}\cos\bar{\omega}t = \frac{P_0}{m}\sin(\bar{\omega}t)$$

Equating coefficients of $\sin\bar{\omega}t$ and $\cos\bar{\omega}t$ separately,

$$(\omega^2 - \bar{\omega}^2)C - 2\xi\omega\bar{\omega}D = \frac{P_0}{m}$$

$$2\xi\omega\bar{\omega}C + (\omega^2 - \bar{\omega}^2)D = 0$$

Dividing by ω^2,

$$\left(1 - \frac{\bar{\omega}^2}{\omega^2}\right)C - 2\xi\frac{\bar{\omega}}{\omega}D = \frac{P_0}{m\omega^2} = \frac{P_0}{K}$$

$$\frac{2\xi\bar{\omega}}{\omega}C + \left(1 - \frac{\bar{\omega}^2}{\omega^2}\right)D = 0$$

or $$(1-\beta^2)C - 2\xi\beta D = \frac{P_0}{K}$$
$$2\xi\beta C + (1-\beta^2)D = 0$$

$\{\because \beta = \frac{\bar{\omega}}{\omega}\}$.

On solving,

$$\therefore C = \frac{P_0}{K}\,\frac{(1-\beta^2)}{(1-\beta^2)^2 + (2\xi\beta)^2}$$

$$\& \; D = \frac{P_0}{K} \times \frac{-2\xi\beta}{(1-\beta^2)^2 + (2\xi\beta)^2}$$

Hence, Complete solution is:-

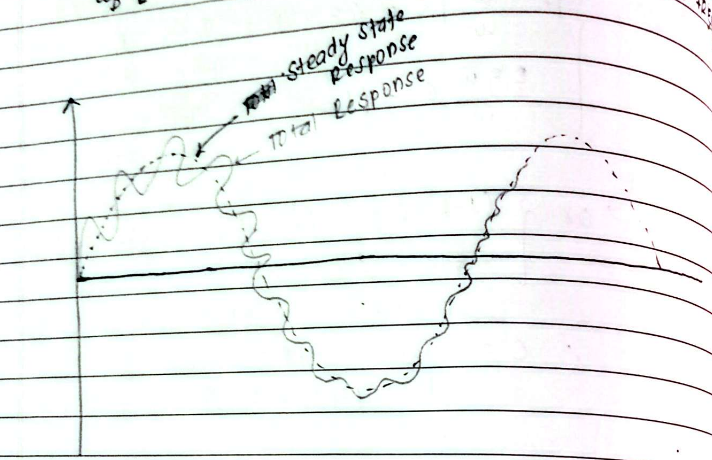
$$u(t) = u_c(t) + u_p(t)$$

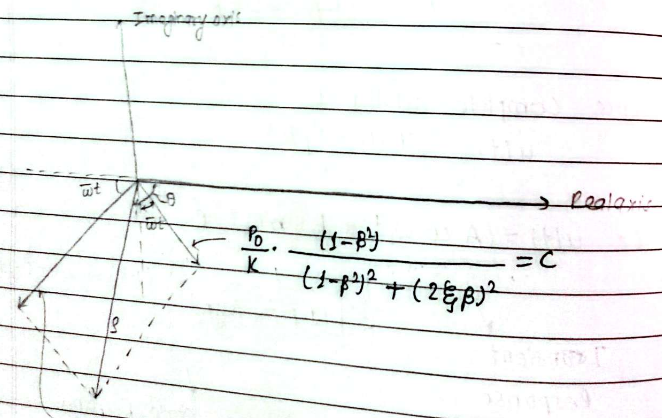
i.e. $$u(t) = \underbrace{(A\cos\omega_D t + B\sin\omega_D t)e^{-\xi\omega t}}_{\text{Transient Response (dependent on initial condition)}} + \underbrace{\frac{P_0}{K}\left\{\frac{1}{(1-\beta^2)^2 + (2\xi\beta)^2}\right\}\left\{(1-\beta^2)\sin\bar{\omega}t - 2\xi\beta\cos\bar{\omega}t\right\}}_{\text{Steady State Response (dependent on loading only)}} \quad \text{—(6)}$$

Using initial condition at $t=0$, $u(t) = u(0)$ & $\dot{u}(t) = \dot{u}(0)$

$$A = u(0) + \frac{P_0}{K} \cdot \frac{2\xi\beta}{(1-\beta^2)^2 + (2\xi\beta)^2}$$

& $$B = \frac{1}{\omega_D}\left[\dot{u}(0) + \xi\omega\left\{u(0) + \frac{P_0}{K}\cdot\frac{2\xi\omega}{(1-\beta^2)^2+(2\xi\beta)^2}\right\} + \frac{P_0}{K}\left\{\frac{\bar{\omega}(1-\beta^2)}{(1-\beta^2)^2+(2\xi\beta)^2}\right.\right.$$





$$\frac{P_0}{K} \cdot \frac{(1-\beta^2)}{(1-\beta^2)^2 + (2\xi\beta)^2} = C$$

$$-D = \frac{P_0}{K} \cdot \frac{2\xi\beta}{(1-\beta^2)^2 + (2\xi\beta)^2}$$

$$\rho = \frac{P_0}{K}\left[(1-\beta^2)^2 + (2\xi\beta)^2\right]^{-1/2} \quad —(7)$$

$$\theta = \tan^{-1}\left\{\frac{2\xi\beta}{(1-\beta^2)}\right\} \quad —(8)$$

$\theta \to$ phase angle by which the response lags behind the applied load.

$$\therefore\ u(t) = \rho \sin(\bar{\omega}t - \theta) \quad —(9)$$

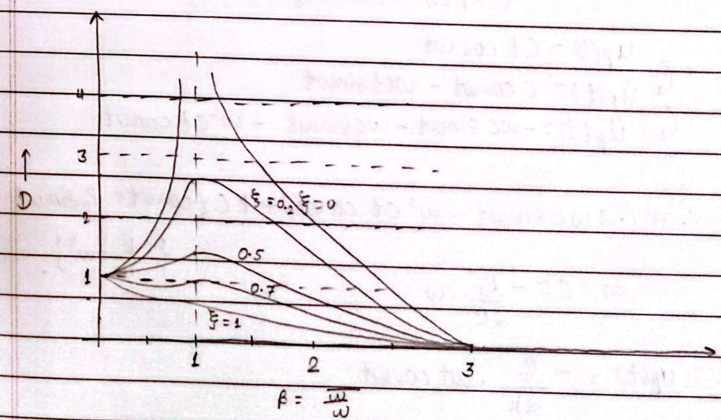
- Dynamic magnification factor (D):

$$D = \frac{\rho}{P_0/K} = \left[(1-\beta^2)^2 + (2\xi\beta)^2\right]^{-1/2}$$

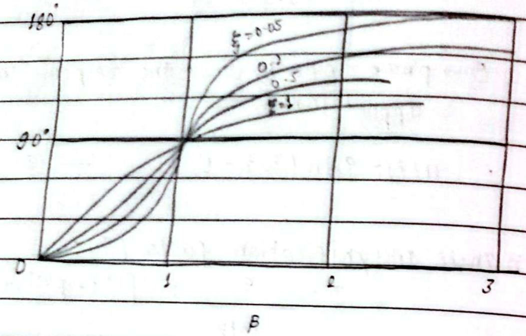
- $$\left[\rho = \frac{P_0}{K} \cdot D\right]$$

$u(t) = \rho \sin(\bar{\omega}t - \theta)$
$u(t) = \frac{P_0}{K} D \sin(\bar{\omega}t - \theta)$
} In terms of Dynamic Magnification factor

$$D = \frac{\rho}{P_0/K} = \left[(1-\beta^2)^2 + (2\xi\beta)^2\right]^{-1/2}$$ } ve.



$$\theta = \tan^{-1}\left\{\frac{2\xi\beta}{(1-\beta^2)}\right\}$$



Resonant Response — forced vibration
[Undamped case]

$$u(t) = A\cos\omega t + B\sin\omega t + \frac{P_0}{K}\cdot\frac{1}{(1-\beta^2)}\sin\bar{\omega} t \quad \text{—(1)}$$

If $\beta = \frac{\bar{\omega}}{\omega} = 1 \Rightarrow \bar{\omega} = \omega \rightarrow$ Resonance

$$m\ddot{u}(t) + k u(t) = P_0 \sin\bar{\omega} t$$

The solution:

$$\cancel{\{u_p(t) = C\sin\bar{\omega}t\}}$$

$$u_p(t) = Ct\cos\omega t$$

$$\dot{u}_p(t) = C\cos\omega t - \omega Ct\sin\omega t$$

$$\ddot{u}_p(t) = -\omega C\sin\omega t - \omega C\sin\omega t - \omega^2 Ct\cos\omega t$$

$$\therefore m(-2\omega C\sin\omega t - \omega^2 Ct\cos\omega t) + kCt\cos\omega t = P_0\sin\omega t$$

$$\Rightarrow C = -\frac{P_0}{2K}\cdot\omega \qquad \left\{\frac{k}{m} = \omega^2\right\}.$$

$$\therefore u_p(t) = -\frac{P_0}{2K}\cdot\omega t\cos\omega t$$

The complete solution will be:

$$u(t) = A\cos\omega t + B\sin\omega t - \frac{P_0}{2K}\omega t\cos\omega t$$

$$\therefore \dot{u}(t) = -\omega A\sin\omega t + \omega B\cos\omega t - \frac{P_0}{2K}\omega\cos\omega t + \frac{P_0}{2K}\omega^2 t\sin\omega t$$

At, $t=0$, $u(t) = u(0)$, $\dot{u}(t) = \dot{u}(0)$; we have

$$A = u(0)$$

$$\& \quad B = \frac{\dot{u}(0)}{\omega} + \frac{P_0}{2K}$$

If initial condition is at rest at origin,

$$A = u(0) = 0 \quad \& \quad B = \frac{P_0}{2K}$$

$$\boxed{\therefore u(t) = -\frac{P_0}{2K}(\omega t\cos\omega t - \sin\omega t)}$$ ← gives resonant response in undamped condition

Resonant Response : (Damped)

Eqn of motion:

$$m\ddot{u}(t) + c\dot{u}(t) + ku(t) = P_0 \sin\bar{\omega}t$$

and Response,

$$u(t) = (A\cos\omega_D t + B\sin\omega_D t)e^{-\xi\omega t} + C\sin\bar{\omega}t + D\cos\bar{\omega}t$$

when, $\omega = \bar{\omega}$ i.e. $\beta = 1$

$$C = \frac{P_0}{K} \cdot \frac{(1-\beta^2)}{(1-\beta^2)^2 + (2\xi\beta)^2} = 0$$

$$D = \frac{P_0}{K} \cdot \frac{(-2\xi\beta)}{(1-\beta)^2 + (2\xi\beta)^2} = -\frac{P_0}{K} \cdot \frac{1}{2\xi}$$

$$\therefore u(t) = (A\cos\omega_D t + B\sin\omega_D t)e^{-\xi\omega t} - \frac{P_0}{K} \cdot \frac{1}{2\xi}\cos\omega t$$

At, $t = 0$; $u(t) = u(0) = 0$

$\dot{u}(t) = \dot{u}(0) = 0$

$$\boxed{A = \frac{P_0}{2\xi K}}$$

Differentiating,

$$\dot{u}(t) = (-\omega_D A\sin\omega_D t + \omega_D B\cos\omega_D t)e^{-\xi\omega t} - \xi\omega e^{-\xi\omega t}(A\cos\omega_D t + B\sin\omega_D t) + \frac{P_0}{K} \cdot \frac{1}{2\xi}\omega\sin\omega t$$

or, $0 = -0 + \omega_D B e^{-0} - \xi\omega . A$

or $B = \frac{\xi\omega A}{\omega_D} = \frac{P_0}{2\xi k} \cdot \frac{\xi\omega}{\omega_D}$

or $\boxed{B = \frac{P_0 \omega}{2k\omega_D} = \frac{P_0}{2K} \cdot \frac{1}{\sqrt{1-\xi^2}}}$

$$\left\{\because \frac{\omega}{\omega_D} = \frac{1}{\sqrt{1-\xi^2}}\right\}$$

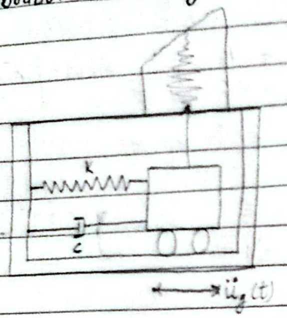
Finally on substituting

$$u(t) = \left\{\frac{P_0}{2\xi K}\cos\omega_D t + \frac{P_0}{2K} \cdot \frac{1}{\sqrt{1-\xi^2}}\sin\omega_D t\right\}e^{-\xi\omega t} - \frac{P_0}{2K\xi}\cos\omega t$$

$$\therefore u(t) = \frac{P_0}{2\xi K}\left[\left(\cos\omega_D t + \frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_D t\right)e^{-\xi\omega t} - \cos\omega t\right]$$

Vibration Measuring Instrument (Seismic Instrument)

Relative displacement Record



Seismometer can measure both
- acceleration → accelerometer
- displacement → displacement meter

Interms of acceleration: $\ddot{u}_g(t) = \ddot{u}_{g_0} \sin\bar{\omega} t$ --- (i)

Equation of motion:

$m\ddot{u}(t) + c\dot{u}(t) + ku(t) = P_{eff}(t)$ [-ve : force opposite direction of $\ddot{u}_g(t)$]

$= -m\ddot{u}_g(t)$

$= -m\, \ddot{u}_{g_0} \sin\bar{\omega} t$

Solution,

Steady state response (Particular solution)

$u(t) = \dfrac{m\ddot{u}_{g_0}}{k} \cdot D\sin(\bar{\omega} t - \theta)$

As, for $p(t) = P_0 \sin\bar{\omega} t$, $u(t) = \dfrac{P_0}{k} \cdot D\sin(\bar{\omega} t - \theta)$

where, D = Dynamic Magnification factor

$D = [(1-\beta^2)^2 + (2\xi\beta)^2]^{-1/2}$

Interms of displacement:

$u_g(t) = u_{g_0} \sin\bar{\omega} t$

$\ddot{u}_g(t) = -\bar{\omega}^2 u_{g_0} \sin\bar{\omega} t$

Equation of motion in this case,

$m\ddot{u}(t) + c\dot{u}(t) + ku(t) = -m\ddot{u}_g(t)$

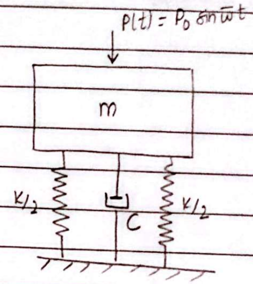
$= -m\bar{\omega}^2 u_{g_0} \sin\bar{\omega} t$

∴ The solution,

$u(t) = \dfrac{m\bar{\omega}^2 u_{g_0}}{k} \cdot D\sin(\bar{\omega} t - \theta)$

$\therefore u(t) = \beta^2 u_{g_0}\, D\sin(\bar{\omega} t - \theta)$

Vibration Isolation [application of harmonic loading].



Steady state Response:

$u(t) = \dfrac{P_0}{k} D\sin(\bar{\omega} t - \theta)$ --- (i)

where, $D = \dfrac{\rho}{P_0/k} = \dfrac{\rho}{P_0/k} = [(1-\beta^2)^2 + (2\xi\beta)^2]^{-1/2}$

(ρ → amplitude (displacement); P_0/k → static displacement)

Spring property: $f_S = k\,u(t) = P_0 D\sin(\bar{\omega} t - \theta)$ → Force exerted by elastic spring.

Damping property: $f_D = c\dot{u}(t) = c\,\dfrac{P_0}{k} \cdot D\bar{\omega}\cos(\bar{\omega} t - \theta)$

$\therefore f_D = 2\xi\beta P_0 D\cos(\bar{\omega} t - \theta)$ → force exerted to base by damper (dash pot)

$\because \xi = \dfrac{c}{2m\omega} \Rightarrow 2\xi\omega = \dfrac{c}{m}$

$k/m = \omega^2$

The maximum value of force transmitted to base

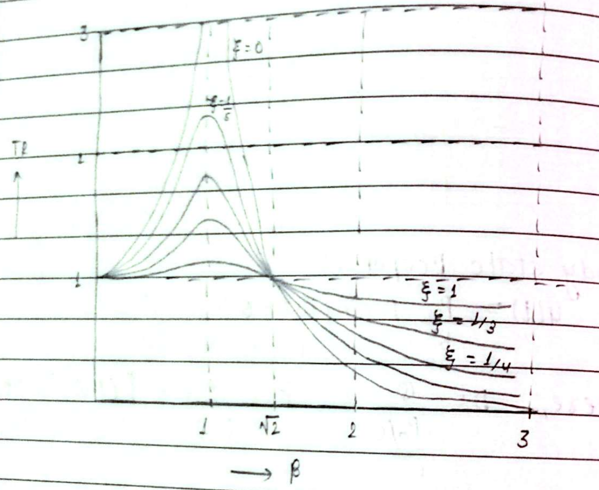
$$f_{max} = [f_{s,max}^2 + f_{D,max}^2]^{1/2}$$

$$= [(P_0 D)^2 + (2\xi\beta P_0 D)^2]^{1/2}$$

$$f_{max} = P_0 D\sqrt{1+(2\xi\beta)^2}$$

* Transmissibility Ratio (T.R.)

$$TR = \frac{f_{max}}{P_0} = D\sqrt{1+(2\xi\beta)^2}$$



$\beta = \frac{\bar{\omega}}{\omega}$, $\xi = \frac{c}{c_c}$

$\beta = \frac{\bar{\omega}}{\omega} = \frac{\bar{\omega}}{\sqrt{k/m}}$

→ β value increases, less transfer of force with decreasing ξ, But $\beta\sqrt{2}$ → else transmission is high i.e. आएको force भन्दा धेरै transfer भएको

$\beta > \sqrt{2}$, base isolation

Energy dissipated by Damping

- Consider steady state motion of a SDOF with

$$P(t) = P_0 \sin\bar{\omega}t$$

$$\left[\begin{aligned} u(t) &= \rho\sin(\bar{\omega}t - \theta) \\ \dot{u}(t) &= \rho\bar{\omega}\cos(\bar{\omega}t - \theta) \end{aligned}\right]$$

- The energy dissipated by viscous damping per cycle of harmonic vibration,

$$E_d = \int f_d \cdot du$$

$$= \int_0^{2\pi/\bar{\omega}} c\dot{u}\dot{u}\,dt$$

$$= \int_0^{2\pi/\bar{\omega}} c\,\dot{u}^2 dt$$

$$= \int_0^{2\pi/\bar{\omega}} c\,[\rho\bar{\omega}\cos(\bar{\omega}t - \theta)]^2 dt$$

$$= c\left\{\bar{\omega}^2\rho^2\frac{\pi}{\bar{\omega}}\right\}$$

$$= \cancel{\pi c \bar{\omega}}\ \pi c\bar{\omega}\rho^2$$

$$= \pi(2m\omega\xi)\bar{\omega}\rho^2$$

$$= 2\pi\xi\left(\frac{k}{\omega}\right)\bar{\omega}\rho^2$$

$$= 2\pi\xi k\cdot\beta\cdot\rho^2$$

$$[E_d = 2\pi\xi k\beta\rho^2] \text{ ---(i)}$$

- In steady state vibration, the energy input to the system due to applied load is

$$E_I = \int p(t)\cdot du \qquad \{du = \dot{u}\,dt\}.$$

$$= \int_0^{2\pi/\bar{\omega}} (P_0\sin\bar{\omega}t)\cdot\dot{u}\,dt$$

$$= \int_0^{(2\pi/\bar{\omega})} [P_0 \sin \bar{\omega} t] \, [\bar{\omega} \rho \cos(\bar{\omega} t - \theta)] \, dt$$

On solving,

$$\left[E_I = \pi P_0 \rho \sin\theta\right]$$

But, $\theta = \tan^{-1}\left(\frac{2\pi\beta}{1-\beta^2}\right)$

$$\tan\theta = \frac{2\pi\beta}{1-\beta^2}$$

$$\therefore \sin\theta = \frac{2\pi\beta/(1-\beta^2)}{\sqrt{\left\{\left(\frac{2\pi\beta}{(1-\beta^2)}\right)^2 + 1\right\}}} \quad \frac{2\pi\beta}{\sqrt{(2\pi\beta)^2 + (1-\beta^2)^2}}$$

we have, on substituting,

$$E_I = 2\pi \xi K \beta \rho^2 \quad ----(ii)$$

Now, $\left[E_D = E_I\right]$.

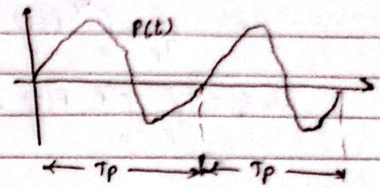
Forced vibration Response to periodic loading

⇒ Periodic function: $f(x) = f(x + \Delta x)$

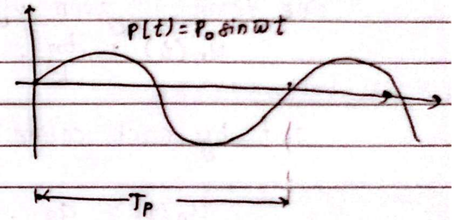
'or'

$P(t) = P(t + n T_p)$

$n = 1, 2, 3, \ldots$



⇒ Harmonic loading → sine or cosine function



⇒ Fourier series:

$$P(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n}{T_p} t\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n}{T_p} t\right)$$

where, T_p = period of excitation load function.

$$\therefore P(t) = a_0 + a_1 \cos\bar{\omega}_1 t + a_2 \cos\bar{\omega}_2 t + \ldots$$
$$+ b_1 \sin\bar{\omega}_1 t + b_2 \sin\bar{\omega}_2 t + \ldots$$

$P(t)$ → summing up series of harmonic function

then, $P(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\bar{\omega} t) + \sum_{n=1}^{\infty} b_n \sin(n\bar{\omega} t)$

where, $\omega_1 = \frac{2\pi}{T_p}$ & $\omega_n = n\omega_1$

i.e. $\omega_2 = 2\omega_1$

$\omega_3 = 3\omega_1$

where $a_0 = \frac{1}{T_p}\int_0^{T_p} p(t)\,dt$

$$a_n = \frac{2}{T_p}\int_0^{T_p} p(t)\cos\left(\frac{2n\pi}{T_p}\cdot t\right)dt$$

$$b_n = \frac{2}{T_p}\int_0^{T_p} p(t)\sin\left(\frac{2n\pi}{T_p}\cdot t\right)dt$$

- Steady state response in an undamped SDOF by each sine term is given by

$$u_n(t) = \frac{b_n}{k}\cdot\frac{1}{(1-\beta_n^2)}\sin\bar{\omega}t$$

- and by each cosine term is given by

$$u_n(t) = \frac{a_n}{k}\cdot\frac{1}{(1-\beta_n^2)}\cos(n\bar{\omega}t)$$

and due to constant term a_0 is given by,

$$u_0 = \frac{a_0}{k}$$

- Hence, the periodic response is

$$u(t) = \frac{a_0}{k} + \sum_{n=1}^{\infty}\frac{a_n}{k}\cdot\frac{1}{(1-\beta_n^2)}\cos(n\bar{\omega}_1 t) + \sum_{n=1}^{\infty}\frac{b_n}{k}\cdot\frac{1}{(1-\beta_n^2)}\sin(n\bar{\omega}_1 t)$$

$$u(t) = \frac{1}{k}\left[a_0 + \sum_{n=1}^{\infty}\frac{1}{(1-\beta_n^2)}\left\{a_n\cos(n\bar{\omega}_1 t) + b_n\sin(n\bar{\omega}_1 t)\right\}\right]$$

× Damped SDOF system:

→ Response due to sine terms:

$$u_n(t) = \frac{b_n}{k} * \frac{1}{[(1-\beta_n^2)+(2\xi\beta_n)^2]} * \left[(1-\beta_n^2)\sin(n\bar{\omega}_1 t) - 2\xi\beta_n\cos(n\bar{\omega}_1 t)\right]$$

→ Due to cosine term:

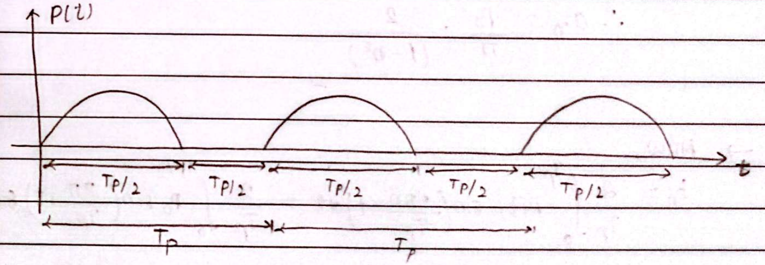
$$u_n(t) = \frac{a_n}{k}\cdot\frac{1}{[(1-\beta_n^2)+(2\xi\beta_n)^2]} * \left[2\xi\beta_n\sin(n\bar{\omega}_1 t) + (1-\beta_n^2)\cos(n\bar{\omega}_1 t)\right]$$

→ Hence, the total periodic response in damped system:

$$u(t) = \frac{1}{k}\left[a_0 + \sum_{n=1}^{\infty}\frac{1}{\{(1-\beta_n^2)+(2\xi\beta_n)^2\}}\left\{[a_n\cdot 2\xi\beta_n + b_n(1-\beta_n^2)]\sin(n\bar{\omega}_1 t) + [a_n(1-\beta_n^2) - b_n 2\xi\beta_n]\cos(n\bar{\omega}_1 t)\right\}\right]$$

Example:

- Half-sine function



$$p(t) = P_0\sin\left(\frac{2\pi}{T_p}\cdot t\right)$$

Solution:

$$a_0 = \frac{1}{T_p}\int_0^{T_p} p(t)\,dt$$

$$= \frac{1}{T_p}\int_0^{T_p/2} P_0\sin\left(\frac{2\pi}{T_p}\cdot t\right)dt = -\frac{P_0}{T_p} * \frac{T_p}{2\pi}\left[\cos\frac{2\pi}{T_p}\right]_0^{T_p/2}$$

$$a_n = \frac{2}{T_p}\int_0^{T_p} p(t)\cos\left(\frac{2\pi n}{T_p}.t\right)dt$$

$$= \frac{2}{T_p}\int_0^{T_p/2} P_0 \sin\left(\frac{2\pi}{T_p}.t\right)\cos\left(\frac{2\pi n}{T_p}t\right)dt$$

$$= \frac{P_0}{T_p}\int_0^{T_p/2}\left[\sin\left\{(1+n)\frac{2\pi t}{T_p}\right\} + \sin\left\{(1-n)\frac{2\pi t}{T_p}\right\}\right]dt$$

$$= \frac{P_0}{T_p}\left[-\frac{\cos\{(1+n)\frac{2\pi}{T_p}t\}}{(1+n)\frac{2\pi}{T_p}} - \frac{\cos\{(1-n)\frac{2\pi}{T_p}.t\}}{(1-n)\frac{2\pi}{T_p}}\right]_0^{T_p/2}$$

$$= \frac{P_0}{2\pi}\left[\frac{-\cos(1+n)\pi + 1}{(1+n)} + \frac{-\cos(1-n)\pi + 1}{(1-n)}\right]$$

For all odd values of n,

$a_n = 0$

$$\left[a_0 = P_0/\pi\right]$$

and for even value of n

$$a_n = \frac{P_0}{2\pi}\left(\frac{2}{1+n} + \frac{2}{1-n}\right)$$

$$\therefore a_n = \frac{P_0}{\pi}\cdot\frac{2}{(1-n^2)}$$

→ Now,

$$b_n = \frac{2}{T_p}\int_0^{T_p} P(t).\sin\left(\frac{2\pi n}{T_p}.t\right)dt = \frac{2}{T_p}\int_0^{T_p/2} P_0\sin\left(\frac{2\pi}{T_p}.t\right)\sin\left(\frac{2\pi n}{T_p}t\right)dt$$

$$= \begin{cases} P_0/2 & \text{for } n=1 \\ 0 & \text{all other } n \end{cases}$$

→ $\therefore$

$$P(t) = \frac{P_0}{\pi} + \frac{P_0}{2}\sin(\bar{\omega}_1 t) - \underbrace{\frac{2}{3}\cdot\frac{P_0}{\pi}\cos(2\bar{\omega}_1 t)}_{n=2} - \underbrace{\frac{2}{15}\frac{P_0}{\pi}\cos(4\bar{\omega}_1 t)}_{n=3}$$

$$-\frac{2}{35}\cdot\frac{P_0}{\pi}\cos(6\bar{\omega}_1\theta) - \dots$$

$$\therefore P(t) = \frac{P_0}{\pi}\left[1 + \frac{\pi}{2}\sin\bar{\omega}_1 t - \frac{2}{3}\cos 2\bar{\omega}_1 t - \frac{2}{15}\cos 4\bar{\omega}_1 t - \frac{2}{35}\cos 6\bar{\omega}_1 t - \dots\right]$$

where, $\bar{\omega}_1 = \frac{2\pi}{T_p}$

→ Undamped response:

For special case, if $\frac{T_p}{T} = \frac{4}{3}$; $\frac{\bar{\omega}}{\omega} = \beta_1 = \frac{3}{4}$

$$\beta_2 = \frac{2\bar{\omega}_1}{\omega} = 3/2$$

Response,

$$u(t) = \frac{P_0}{K\pi}\left[1 + \frac{8\pi}{7}\sin\bar{\omega}_1 t + \frac{8}{15}\cos 2\bar{\omega}_1 t + \frac{1}{60}\cos 4\bar{\omega}_1 t + \dots\right]$$

Fourier series in exponential form (complex loading)

$$F(x) = a_0 + \sum_{n=1}^{\infty} \{a_n \cos nx + b_n \sin nx\}$$

$$P(t) = a_0 + \sum_{n=1}^{\infty} \{a_n \cos(n\omega_p t) + b_n \sin(n\omega_p t)\}$$

Now, using euler's relationship,

$e^{i\theta} = \cos\theta + i\sin\theta$ or, $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$; $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$

$e^{-i\theta} = \cos\theta - i\sin\theta$; $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2}$; $\sin x = \dfrac{e^{ix} - e^{-ix}}{2}$

After substituting for sine & cosine terms,

$$F(x) = a_0 + \sum_{n=1}^{\infty}\left[\frac{a_n}{2}\{e^{inx} + e^{-inx}\} + \frac{b_n}{2i}\{e^{inx} - e^{-inx}\}\right]$$

$$\therefore F(x) = a_0 + \sum_{n=1}^{\infty}\left\{\left(\frac{a_n - ib_n}{2}\right)e^{inx} + \left(\frac{a_n + ib_n}{2}\right)e^{-inx}\right\}$$

$\left[\because \frac{1}{i} = -i\right]$.

Now, writing,

$a_0 = C_0$; $\dfrac{a_n - ib_n}{2} = C_n$ and, $\dfrac{a_n + ib_n}{2} = K_n$

$$\therefore F(x) = C_0 + \sum_{n=1}^{\infty}\{C_n e^{inx} + K_n e^{-inx}\} \quad \text{---(i)}$$

where,

→ $C_n = \dfrac{1}{2}(a_n - ib_n) = \dfrac{1}{T_p}\displaystyle\int_0^{T_p} F(x)\{\cos nx - i\sin nx\}dx$

$$C_n = \frac{1}{T_p}\int_0^{T_p} F(x)e^{-inx}dx.$$

→ $K_n = \dfrac{1}{n}(a_n + ib_n) = \dfrac{1}{T_p}\displaystyle\int_0^{T_p} F(x)\{\cos nx + i\sin nx\}dx$

$$K_n = \frac{1}{T_p}\int_0^{T_p} F(x).\,e^{inx}dx.$$

writing $K_n = C_{-n}$

Thus,

$$\boxed{F(x) = \sum_{n=-\infty}^{+\infty} C_n e^{-inx}.dx}$$ ~~same as eq^n (i)~~

↳ Fourier series in exponential form (complex form)

Response to exponential loading (complex loading)

The differential equation of motion

$$m\ddot{u}(t) + c\dot{u}(t) + k\,u(t) = P(t)$$

Let, $P(t) = e^{i\bar{\omega}t}$

So, the steady-state response of exponential/complex loading is given by:-

$$u(t) = H(\bar{\omega}).e^{i\bar{\omega}t}$$

where, $H(\bar{\omega})$ = complex frequency response function.

$$\therefore \dot{u}(t) = i\bar{\omega}.H(\bar{\omega})e^{i\bar{\omega}t}$$

$$\& \ \ddot{u}(t) = \underbrace{i^2}_{-1}\bar{\omega}^2.H(\bar{\omega})e^{i\bar{\omega}t}$$

Substituting in eq^n of motion:-

or, $-m\bar{\omega}^2 H(\bar{\omega})e^{i\bar{\omega}t} + C i\bar{\omega}H(\bar{\omega})e^{i\bar{\omega}t} + k.H(\bar{\omega})e^{i\bar{\omega}t} = e^{i\bar{\omega}t}$

Now,

Introducing, $\xi = \dfrac{c}{2m\omega}$ and $\beta = \dfrac{\bar{\omega}}{\omega}$; $\dfrac{k}{m} = \omega^2$ & $\dfrac{c}{k} = \dfrac{2m\omega\xi}{m\omega^2}$

$\left[\dfrac{c}{k} = \dfrac{2\xi}{\omega}\right]$

$$\boxed{\therefore H(\bar{\omega}) = \frac{1}{K(-\beta^2 + 2\xi\beta i + 1)}}$$

Hence, the steady state response to exponentially complex loading is:

$$u(t) = H(\bar{w}) e^{i\bar{w}t} = \frac{1}{k(-\beta^2 + 2\xi\beta i + 1)} e^{i\bar{w}t}$$

Response to periodic loading expressed as fourier series loading in exponential form

The differential eq^n of motion is:

$$m\ddot{u}(t) + c\dot{u}(t) + k u(t) = P(t)$$

where, $P(t) = \sum_{n=-\infty}^{+\infty} C_n e^{in\bar{w}t}$

$$C_n = \frac{1}{T_p}\int_0^{T_p} P(t) e^{i\bar{w}t} dt$$

As we know, the steady state response to exponential complex loading is

$$u(t) = H(\bar{w}) e^{i\bar{w}t} = \frac{1}{k(-\beta^2 + 2\xi\beta i + 1)} e^{i\bar{w}t}$$

Hence, the steady-state response to periodic loading expressed as fourier series loading in exponential form/complex form:

$$\boxed{u(t) = \sum_{-\infty}^{\infty} H(n\bar{w}) C_n e^{in\bar{w}t}}$$

where, $H(n\bar{w}) = \frac{1}{k(-n^2\beta^2 + 2\xi n\beta i + 1)}$

$$C_n = \frac{1}{T_p}\int_0^{T_p} P(t) e^{in\bar{w}t} dt \quad \left\{\because T_p = \frac{2\pi}{\bar{w}}\right\}$$

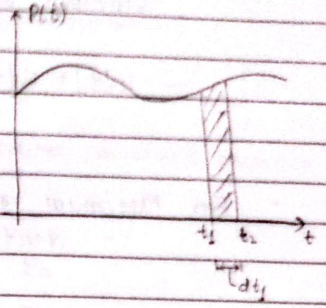
Impulsive loading

→ consist of single principle impulse

→ very short time (duration)

→ No time for dissipation of energy, so that no damping will occur (i.e. damping property doesn't work {during loading period}.

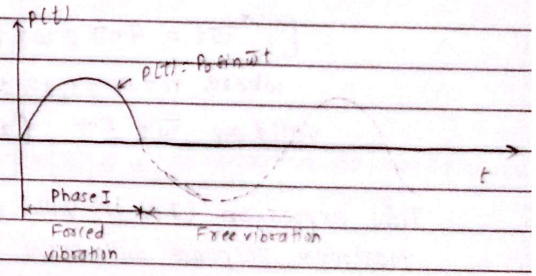
→ eg: Bomb Blast



$[t_2 = t_1 + dt_1]$

Response to Half-sine-wave impulsive:

Assuming the system starts from rest



Phase I: $0 \le t \le t_1$ [Forced vibration]

System starts from rest. Then the total response (transient as well as steady state for undamped forced vibration is given by:

$$u(t) = \frac{P_0}{K} \cdot \frac{1}{1-\beta^2} (\sin \bar{w}t - \beta \sin wt) \quad \text{--- (I1)}$$

Phase II : $t > t_1$ [Free vibration]

Introducing $\bar{t} = (t - t_1) \geq 0$ (new time)

In this case initial displacement/position, velocity & acceleration are ~~not~~ not zero. So, undamped free vibration response is given by:

$$u(\bar{t}) = u(t_1)\cos\omega\bar{t} + \frac{\dot{u}(t_1)}{\omega}\sin\omega\bar{t} \quad \text{-----} (\text{I}_2)$$

- For Maximum response in phase I,

$$\frac{du(t)}{dt} = 0$$

$$\frac{P_0}{k} \cdot \frac{1}{1-\beta^2}\left[\bar{\omega}\cos\bar{\omega}t - \beta\omega\cos\omega t\right] = 0$$

$$\Rightarrow \cos\bar{\omega}t = \cos\omega t$$

$$\left[\therefore \bar{\omega}t = 2n\pi \pm \omega t\right] \text{—} (\text{I}3)$$

where, $n = 0, \pm 1, \pm 2, \pm 3, \ldots$

valid for $\bar{\omega}t \leq \pi$ {as Half-Sine wave}

This expression (I3) is valid only for $\bar{\omega}t \leq \pi$ i.e. maximum response will occur while impulsive load is acting (phase I) when the loading frequency approaches the free vibration frequency i.e. when $\bar{\omega} \to \omega$, the time of maximum response will be given by substituting $n = 1$ and & using $-ve$ sign in eq^n (I3),

$$\bar{\omega}t = 2\pi - \omega t$$

or, $(\bar{\omega} + \omega)t = 2\pi$

or, $$\left[t = \frac{2\pi}{\bar{\omega}+\omega} = \frac{2}{\frac{\omega}{\pi} + \frac{\bar{\omega}}{\pi}} = \frac{2}{\frac{1}{t_1} + \frac{2}{T}}\right]$$

$t_1 \to$ period of half-sine wave.

$T \to$ time-period of system (structure)

$$\left[t = \frac{2}{\frac{1}{t_1} + \frac{2}{T}} \quad \text{valid for } t \leq t_1\right]$$

or, $$\frac{2}{\left(\frac{1}{t_1} + \frac{2}{T}\right)} \leq t_1$$

or, $$2 \leq \frac{2t_1}{T} + 1$$

or, $$\boxed{\frac{t_1}{T} \geq \frac{1}{2}}$$ which is condition for max^m response.

Also,

$$\left[\frac{t_1}{T} = \frac{\pi/\bar{\omega}}{2\pi/\omega} = \frac{\omega}{2\bar{\omega}} = \frac{1}{2\beta}\right].$$

So, $$\frac{1}{2\beta} \geq \frac{1}{2}$$

$$\Rightarrow \frac{1}{\beta} \geq 1$$

or, $[\beta \leq 1] \longrightarrow [\bar{\omega} \leq \omega]$ ← condition for max^m response in phase-I

~~So, to~~

So, to have maximum response in phase I (during impulse)

$$\left.\begin{array}{l} \frac{t_1}{T} \geq \frac{1}{2} \\ \text{or, } \beta \leq 1 \\ \text{or, } \bar{\omega} \leq \omega \end{array}\right\}$$

$\therefore$ U_{max} can be obtained by substituting $t = \frac{2\pi}{\bar{\omega}+\omega}$ in eq^n (I1)

$$\therefore u_{max} = \frac{P_0}{K}\cdot\frac{1}{(1-\beta^2)}\left[\sin\left\{\bar{\omega} \times \frac{2\pi}{(\omega+\bar{\omega})}\right\} - \beta \sin\left\{\omega \times \frac{2\pi}{(\omega+\bar{\omega})}\right\}\right]$$

$$\therefore \boxed{u_{max} = \frac{P_0}{K}\cdot\frac{1}{(1-\beta^2)}\left[\sin\left(\frac{2\pi\beta}{1+\beta}\right) - \beta\sin\left(\frac{2\pi}{1+\beta}\right)\right]} \text{ --- (I4)}$$

$$\text{or, } D = \frac{u_{max}}{P_0/K} = \frac{1}{(1-\beta^2)}\left[\sin\left(\frac{2\pi\beta}{1+\beta}\right) - \beta\sin\left(\frac{2\pi}{1+\beta}\right)\right] \text{ --- (I5)}$$

for $\frac{t_1}{T} \geq \frac{1}{2}$

- For maximum response in phase II (after impulse)

i.e. for $\frac{t_1}{T} < \frac{1}{2}$, max^m response will occur at $t = t_1 = \frac{\pi}{\bar{\omega}}$

$$u(t_1) = \frac{P_0}{K}\cdot\frac{1}{(1-\beta^2)}\left[\sin\left\{\bar{\omega}\cdot\frac{\pi}{\bar{\omega}}\right\} - \beta\sin\left\{\omega\cdot\frac{\pi}{\bar{\omega}}\right\}\right].$$

$$\therefore u(t_1) = \frac{P_0}{K}\cdot\frac{1}{(1-\beta^2)}\left[0 - \beta\sin\left(\frac{\pi}{\beta}\right)\right]$$

$$\therefore \dot{u}(t_1) = \frac{P_0}{K}\cdot\frac{1}{(1-\beta^2)}\left[\bar{\omega}\cos(\bar{\omega}t_1) - \beta\cdot\omega\cos(\omega t_1)\right]$$

$$\therefore \dot{u}(t_1) = \frac{P_0}{K}\cdot\frac{\bar{\omega}}{(1-\beta^2)}\left[-1 - \cos(\pi/\beta)\right]$$

$\therefore$ The amplitude of free vibration:-

$$\rho = \left[[u(t_1)]^2 + \left[\frac{\dot{u}(t_1)}{\omega}\right]^2\right]^{1/2} \quad \{I2\}$$

$$= \frac{P_0}{K}\cdot\frac{\beta}{(1-\beta^2)} * 2\cos\left(\frac{\pi}{2\beta}\right) \text{ --- (I6)}$$

$$D = \frac{\rho}{P_0/K} = \frac{2\beta}{(1-\beta^2)}\cdot\cos\left(\frac{\pi}{2\beta}\right)$$

Response to rectangular impulse (step loading)

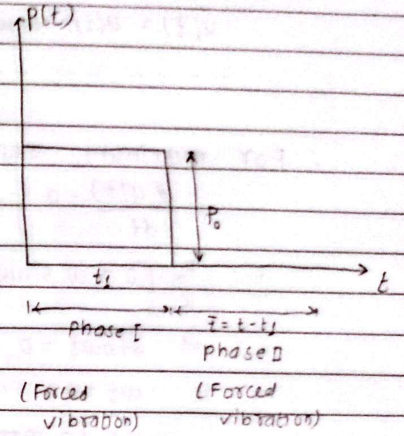
- Phase I (during impulse)

Eq^n of motion

$$m\ddot{u}(t) + k\,u(t) = P_0$$

The particular solution is

$$u_p = \frac{P_0}{K}$$



Also, complimentory solution:-

$$u_c = A\cos\omega t + B\sin\omega t$$

Hence, total response:

$$u(t) = A\cos\omega t + B\sin\omega t + \frac{P_0}{K}$$

$\therefore$ If initial condition is at rest i.e. $u(0) = 0$, $\dot{u}(0) = 0$

$$\therefore u(0) = 0 = A\cos 0 + B\sin 0 + \frac{P_0}{K}$$

$$\Rightarrow A = -\frac{P_0}{K}$$

$$\dot{u}(t) = -A\omega\sin\omega t + B\omega\cos\omega t$$

$$\therefore \dot{u}(0) = 0 = -A\omega\sin(0) + B\omega\cos(0)$$

$$\Rightarrow B = 0$$

Hence, total response during impulse:-

$$\boxed{u(t) = \frac{P_0}{K}\left[1 - \cos\omega t\right]}$$

Phase II (after impulse)
$(\bar{t} = t - t_1 \geq 0)$

The undamped free vibration response in this phase is given by:-

$$\boxed{u(\bar{t}) = u(t_1)\cos\omega\bar{t} + \frac{\dot{u}(t_1)}{\omega}\sin\omega\bar{t}}$$

∴ For maximum response in phase I:

$$\frac{d\,u(t)}{dt} = 0$$

$$\frac{P_0}{K}[0 + \omega\sin\omega t] = 0$$

$$\Rightarrow \sin\omega t = 0$$

$$\Rightarrow \omega t = n\pi \quad \text{where, } n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

$$\therefore t = \frac{n\pi}{\omega}$$

For $n = 1$, $t = \frac{\pi}{\omega} = \frac{T}{2}$

∴ For rectangular impulse (step loading) the maximum response will occur always in phase I if

$$t_1 \geq T/2.$$

$$\therefore u_{max} = \frac{P_0}{K}\left[1 - \cos\omega\cdot\frac{\pi}{\omega}\right] = \frac{2P_0}{K}$$

$$\therefore D = \frac{u_{max}}{P_0/K} = 2$$

For phase II (after impulse)
(Free vibration)

$$\rho = u_{max} = \left\{[u(t_1)]^2 + \left[\frac{\dot{u}(t_1)}{\omega}\right]^2\right\}^{1/2}$$

Now,

$$u(t_1) = \frac{P_0}{K}[1 - \cos\omega t_1]$$

$$\dot{u}(t_1) = \frac{P_0}{K}\omega\sin\omega t_1$$

$$\because \rho = u_{max} = \frac{P_0}{K} * 2\sin\left(\frac{\omega t_1}{2}\right)$$

$$\text{where, } \frac{\omega t_1}{2} = \frac{2\pi}{T}\cdot\frac{t_1}{2} = \frac{\pi t_1}{T}$$

$$\therefore \boxed{\rho = \frac{P_0}{K}\,2\sin\left(\frac{\pi t_1}{T}\right)}$$

$$\therefore D = \frac{\rho}{P_0/K} = 2\sin\left(\frac{\pi t_1}{T}\right)$$

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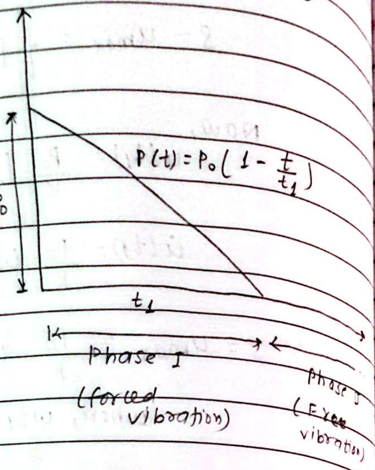
Response to Triangular impulse (Ramp loading)

$P(t) = P_0\left[1 - \frac{t}{t_1}\right]$

Phase I : $(0 \le t \le t_1)$



$u_p = \frac{P_0}{K}\left(1 - \frac{t}{t_1}\right)$

$u_c = A\cos\omega t + B\sin\omega t$

$\therefore$ Total response

$u(t) = u_c + u_p$

$u(t) = A\cos\omega t + B\sin\omega t + \frac{P_0}{K}\left(1 - \frac{t}{t_1}\right)$

Initial condition at rest at origin

i.e. $u(0) = 0$, $\dot{u}(0) = 0$

$\therefore u(0) = 0 = A * 1 + 0 + \frac{P_0}{K}$

$$\left[A = -\frac{P_0}{K}\right]$$

$\dot{u}(t) = -A\omega\sin\omega t + B\omega\cos\omega t + \frac{P_0}{K} * \frac{-1}{t_1}$

$\therefore \dot{u}(0) = -A\omega * 0 + B\omega * 1 - \frac{P_0}{Kt_1} = 0$

$$\left[B = \frac{P_0}{\omega K t_1}\right]$$

$\therefore$ Total solution,

$u(t) = \frac{P_0}{K}\left\{\frac{1}{\omega t_1}\sin\omega t - \cos\omega t + \left(1 - \frac{t}{t_1}\right)\right\}$

Phase II: $t > t_1$ (after impulse)

$\bar{t} = t - t_1 \ge 0$.

The undamped-free vibration response is given by

$u(\bar{t}) = u(t_1)\cos\bar{\omega}t + \frac{\dot{u}(t_1)}{\omega}\sin\omega\bar{t}$

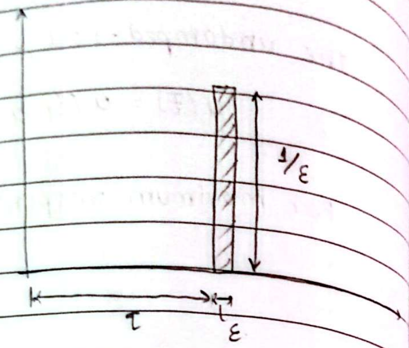
For maximum response

Response to unit impulse

* Unit impulse:

Impulse:

$$I = \int P(t)\,dt$$



When impulse I is equal to unity. Such a force in the limiting case $\varepsilon \to 0$ is called as unit impulse. In the figure shown,

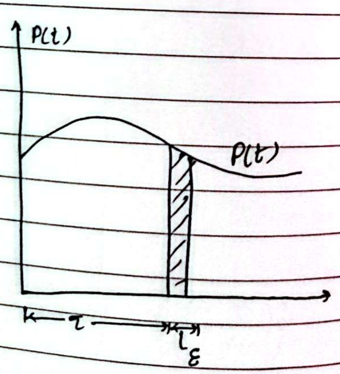
$$P(t) = 1/\varepsilon$$

with time duration ε starting at the time instant $t = \tau$

As $\varepsilon \to 0$, force becomes infinite, however the magnitude of the impulse defined by the time integral of $P(t)$ remains equal to unity.

* Dirac-Delta function

The dirac-delta function denoted by $\delta(t-\tau)$, mathematically defines a unit impulse centered at $t = \tau$.



The Dirac-Delta function has following parameters,

$$\delta(t-\tau) = 0 \quad \text{for all } t \neq \tau$$

$= $ greater than any assumed value for $t = \tau$

$$\int_0^\infty \delta(t-\tau)\,dt = 1$$

$$0 < \tau < \infty$$

$$\therefore \int_0^\infty P(t)\,\delta(t-\tau)\,dt = P(\tau)\ ; \ 0 < \tau < \infty$$

→ According to Newton's second law of motion:

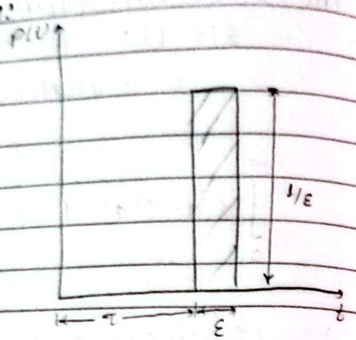
$$\frac{d(m\dot{u})}{dt} = P(t)$$

$$\Rightarrow \quad m\ddot{u} = P(t)$$

$$\therefore \underbrace{\int_{t_1}^{t_2} P(t)\,dt}_{\text{Impulse}} = m(\dot{u}_2 - \dot{u}_1) = \underbrace{m\,\Delta\dot{u}}_{\text{change in momentum}}$$

$$\left[\therefore I = m\,\Delta\dot{u}\right]$$

Response to unit impulse:



$I = P(t)dt$

$I = m\Delta\dot{v}$

or, $\Delta\dot{v} = \frac{I}{m}$

If impulse is unit i.e.

$\Delta\dot{v} = \frac{1}{m}$

The unit impulse at $t=\tau$ imparts the velocity given by:

$\dot{v}(\tau) = \frac{1}{m}$

but displacement prior to/up to impulse:

$v(\tau) = 0$

A unit impulse causes free vibration of the SDOF system due to initial velocity and displacement at $t=\tau$. Hence, response to unit impulse for undamped free vibration is given by:

$$v(t) = v(\tau)\cos\omega\bar{t} + \frac{\dot{v}(\tau)}{\omega}\sin\omega\bar{t}$$

$[\because \bar{t} = t-\tau]$.

$$h(t-\tau) \equiv v(t) = v(\tau)\cos\omega(t-\tau) + \frac{\dot{v}(\tau)}{\omega}\sin\omega(t-\tau)$$

$$h(t-\tau) \equiv v(t) = 0 * \cos\omega(t-\tau) + \frac{1}{m\omega}\sin\omega(t-\tau)$$

$$\therefore \boxed{\underbrace{h(t-\tau)}_{\text{unit impulse response function}} = v(t) = \frac{1}{m\omega}\sin\omega(t-\tau)} \quad ; t \geq \tau$$

for undamped case.

Also, response due to unit impulse for damped free vibration:

$$h(t-\tau) \equiv v(\bar{t}) = e^{-\xi\omega\bar{t}}\left[v(\tau)\cdot\cos\omega_D\bar{t} + \frac{\dot{v}(\tau)+v(\tau)\omega}{\omega_D}\sin\omega_D\bar{t}\right]$$

where, $\omega_D = \omega\sqrt{1-\xi^2}$

or, $$h(t-\tau) \equiv v(\bar{t}) = e^{-\xi\omega\bar{t}}\left[0 + \frac{1/m + 0}{\omega_D} \times \sin\omega_D(t-\tau)\right]$$

$$\therefore h(t-\tau) \equiv v(\bar{t}) = e^{-\xi\omega\bar{t}} * \frac{1}{m\omega_D} * \sin[\omega_D(t-\tau)]; \; t \geq \tau.$$

Also, response due to impulse I;

$$v(t) = I * h(t-\tau) = \frac{I}{m\omega}\sin\omega(t-\tau) \; ; \quad t \geq \tau$$

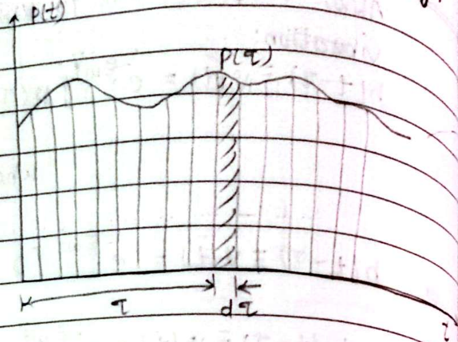
→ for undamped system.

and

$$v(t) = I * h(t-\tau) = \frac{I}{m\omega_D} e^{-\xi\omega(t-\tau)} \cdot \sin\omega_D(t-\tau)$$

→ For damped system.

Response to Arbitrary force/General dynamic loading:



Any force P(t) varying arbitrarily with time can be represented as sequence of infinitesimally short impulse.

The response of linear dynamic system for one of these impulses, the one at time τ of magnitude $I = P(\tau)\cdot d\tau$ is given by:-

unit impulse response function.

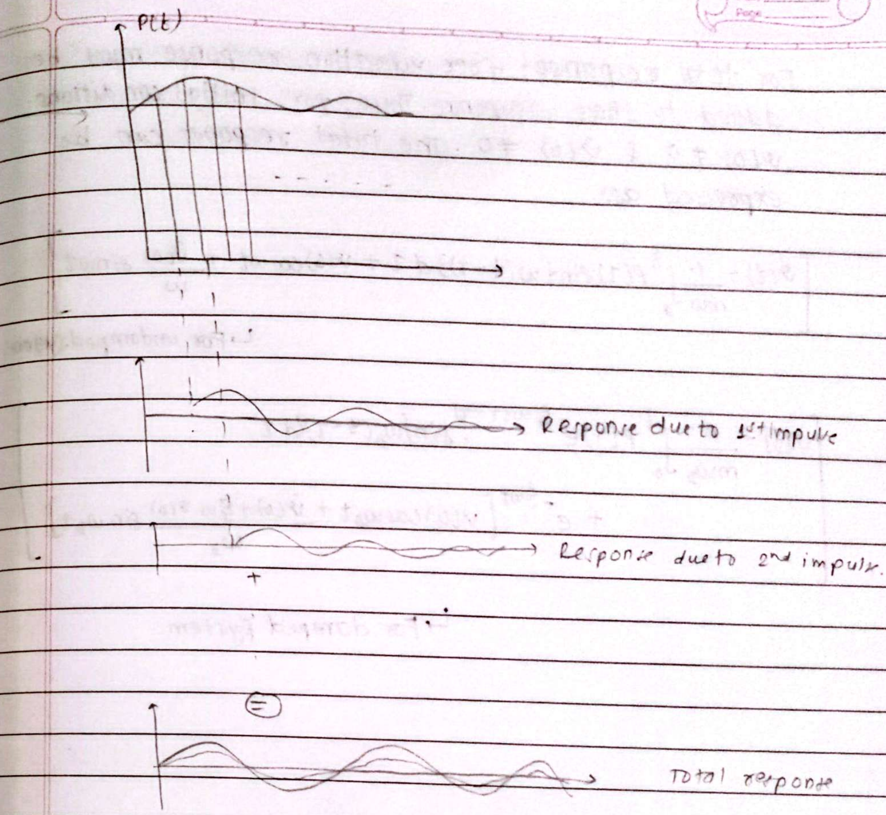
$$dv(t) = I * h(t-\tau)$$

$$\therefore dv(t) = [P(\tau)d\tau] * h(t-\tau) \;;\; t \geq \tau$$

Hence, total response of the system at time 't' is the sum of the responses due to all the impulses upto that time t.

$$\boxed{v(t) = \int_0^t dv(t) = \int_0^t P(\tau)\cdot h(t-\tau)\cdot d\tau}$$

↳ Convolution Integral (Superposition Integral)



→ Now, after substituting for unit impulse response function $h(t-\tau)$ in convolution integral gives

* Duhamel Integral

$$\boxed{v(t) = \frac{1}{m\omega}\int_0^t P(\tau)\sin\{\omega(t-\tau)\}d\tau}$$

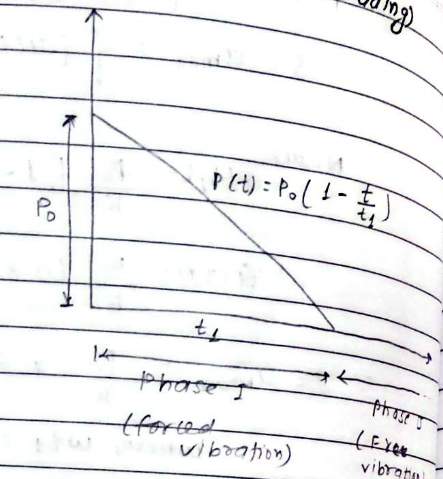
↳ Duhamel Integral (for undamped system)

$$\boxed{v(t) = \frac{1}{m\omega_D}\int_0^t P(\tau)\, e^{-\xi\omega(t-\tau)}\cdot\sin\{\omega_D(t-\tau)\}d\tau}$$

↳ Duhamel Integral (for damped system)

Response to Triangular impulse (Ramp loading)

$P(t) = P_0\left[1 - \frac{t}{t_1}\right]$



Phase I : $(0 \le t \le t_1)$

$\left[u_p = \frac{P(t)}{K}\right]$ $\quad u_p = \frac{P_0}{K}\left(1 - \frac{t}{t_1}\right)$

$u_c = A\cos\omega t + B\sin\omega t$

∴ Total response

$u(t) = u_c + u_p$

$u(t) = A\cos\omega t + B\sin\omega t + \frac{P_0}{K}\left(1 - \frac{t}{t_1}\right)$

Initial condition at rest at origin

i.e. $u(0) = 0$, $\dot{u}(0) = 0$

$\therefore u(0) = 0 = A*1 + 0 + \frac{P_0}{K}$

$$\left[A = -\frac{P_0}{K}\right]$$

$\dot{u}(t) = -A\omega\sin\omega t + B\omega\cos\omega t + \frac{P_0}{K} * \frac{-1}{t_1}$

$\therefore \dot{u}(0) = -A\omega*0 + B\omega*1 - \frac{P_0}{Kt_1} = 0$

$$\left[B = \frac{P_0}{\omega K t_1}\right]$$

∴ Total solution,

$u(t) = +\frac{P_0}{K}\left\{\frac{1}{\omega t_1}\sin\omega t - \cos\omega t + \left(1 - \frac{t}{t_1}\right)\right\}$

Phase II: $t > t_1$ (after impulse)

$\bar{t} = t - t_1 \ge 0$.

The undamped-free vibration response is given by

$u(\bar{t}) = u(t_1)\cos\omega\bar{t} + \frac{\dot{u}(t_1)}{\omega}\sin\omega\bar{t}$

For maximum response

$\dot{u}(t) = \frac{P_0}{K}\left[\frac{1}{t_1}\cos\omega t + \omega\sin\omega t - \frac{1}{t_1}\right] = 0$

At $t = t_1$ or, $(\& \div \text{by } \omega)$ $\quad \frac{\cos\omega t_1}{\omega t_1} + \sin\omega t_1 - \frac{1}{\omega t_1} = 0$

Or $\cos\omega t_1 + \omega t_1\sin\omega t_1 - 1 = 0$

Solving

$\omega t_1 = 2.33$ rad.

As impulsive load,

$\omega t_1 \approx \omega t = 2.33$ rad.

Thus, $t = \frac{2.33}{\omega}$

For maxima to be in phase (I),

$t \le t_1$ i.e. $\frac{2.33}{\omega} \le t_1$

condition

$\frac{2.33*T}{2\pi} \le t_1 \Rightarrow \left[\frac{t_1}{T} \ge 0.371\right]$

point, $\left[t = \frac{2.33}{\omega} = 0.371T\right]$

Thus,

$u_{max} = \frac{P_0}{K}\left[\frac{1}{\omega t_1}\sin\left(\omega*\frac{2.33}{\omega}\right) - \cos\left(\omega*\frac{2.33}{\omega}\right) + \left(1 - \frac{2.33}{\omega t_1}\right)\right]$

$= \frac{P_0}{K}\left[\frac{0.725}{\omega t_1} + 0.688 + 1 - \frac{2.33}{\omega t_1}\right]$

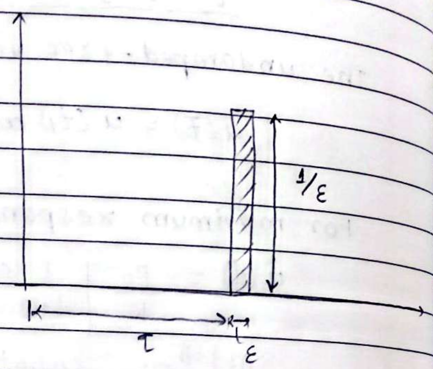
$$\left[u_{max} = \frac{P_0}{K}\left[1.688 - \frac{1.605}{\omega t_1}\right]\right]$$

Response to unit impulse

* Unit impulse:

Impulse:

$$I = \int P(t)\, dt$$



When impulse I is equal to unity. Such a force in the limiting case $\varepsilon \to 0$ is called as unit impulse. In the figure shown,

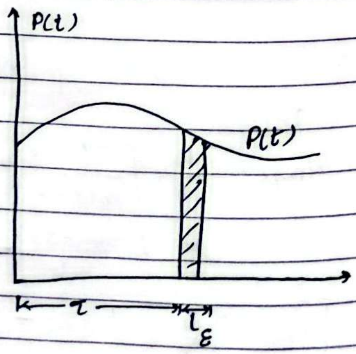
$$P(t) = 1/\varepsilon$$

with time duration ε starting at the time instant $t = \tau$

As $\varepsilon \to 0$, force becomes infinite, however the magnitude of the impulse defined by the time integral of $P(t)$ remains equal to unity.

* Dirac-Delta function

The dirac-delta function denoted by $\delta(t-\tau)$, mathematically defines a unit impulse centered at $t = \tau$.



The Dirac-Delta function has following parameters.

$\delta(t-\tau) = 0$ for all $t \neq \tau$

= greater than any assumed value for $t = \tau$

$$\int_0^\infty \delta(t-\tau)\, dt = 1 \qquad 0 < \tau < \infty$$

$$\therefore \int_0^\infty P(t)\,\delta(t-\tau)\, dt = P(\tau) \; ; \; 0 < \tau < \infty.$$

→ According to Newton's second law of motion:

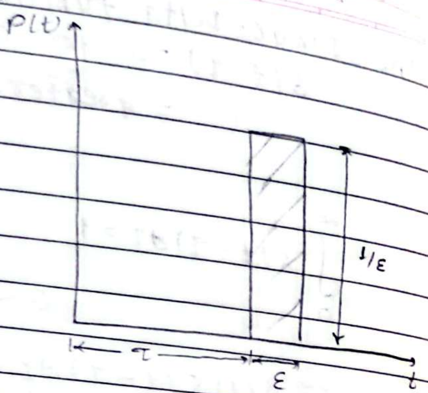
$$\frac{d(m\dot{u})}{dt} = P(t)$$

$$\Rightarrow \quad m\ddot{u} = P(t)$$

$$\therefore \underbrace{\int_{t_1}^{t_2} P(t)\, dt}_{\text{Impulse}} = m(\dot{u}_2 - \dot{u}_1) = \underbrace{m\,\Delta\dot{u}}_{\text{change in momentum}}$$

$$[\therefore I = m\,\Delta\dot{u}]$$

...to unit impulse:



$I = P(t)\,dt$

$I = m\,\Delta\dot{v}$

or, $\Delta\dot{v} = \dfrac{I}{m}$

If impulse is unit i.e.

$\Delta\dot{v} = \dfrac{1}{m}$

The unit impulse at $t = \tau$ imparts the velocity given by:

$\dot{v}(\tau) = \dfrac{1}{m}$

but displacement prior to/up to impulse:

$v(\tau) = 0$

A unit impulse causes free vibration of the SDOF system due to initial velocity and displacement at $t = \tau$. Hence, response to unit impulse for undamped free vibration is given by:

$$v(\bar{t}) = v(\tau)\cos\omega\bar{t} + \frac{\dot{v}(\tau)}{\omega}\sin\omega\bar{t}$$

$[\because \bar{t} = t - \tau]$.

or $h(t-\tau) \equiv v(\bar{t}) = v(\tau)\cos\omega(t-\tau) + \dfrac{\dot{v}(\tau)}{\omega}\sin\omega(t-\tau)$

or. $h(t-\tau) \equiv v(\bar{t}) = 0 \times \cos\omega(t-\tau) + \dfrac{1}{m\omega}\sin\omega(t-\tau)$

$$\therefore \left[\underbrace{h(t-\tau)}_{\text{unit impulse response function}} = v(\bar{t}) = \frac{1}{m\omega}\sin\omega(t-\tau)\right] ; \; t \geq \tau$$

for undamped case.

Also, response due to unit impulse for damped free vibration:

$$h(t-\tau) \equiv v(\bar{t}) = e^{-\xi\omega\bar{t}}\left[v(\tau)\cdot\cos\omega_D\bar{t} + \frac{\dot{v}(\tau) + v(\tau)\omega\xi}{\omega_D}\sin\omega_D\bar{t}\right]$$

where, $\omega_D = \omega\sqrt{1-\xi^2}$

or $h(t-\tau) \equiv v(\bar{t}) = e^{-\xi\omega\bar{t}}\left[0 + \dfrac{1/m + 0}{\omega_D} \times \sin\omega_D(t-\tau)\right]$

$$\therefore h(t-\tau) \equiv v(\bar{t}) = e^{-\xi\omega\bar{t}} \times \frac{1}{m\omega_D} \times \sin[\omega_D(t-\tau)] ; \; t \geq \tau.$$

Also, response due to impulse I;

$v(t) = I \times h(t-\tau) = \dfrac{I}{m\omega}\sin\omega(t-\tau) ; \quad t \geq \tau$

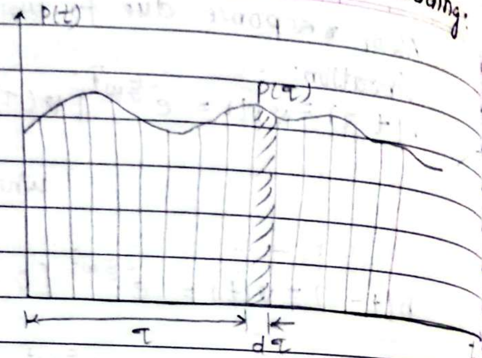
→ for undamped system.

and

$v(t) = I \times h(t-\tau) = \dfrac{I}{m\omega_D} e^{-\xi\omega(t-\tau)} \cdot \sin\omega_D(t-\tau)$

→ For damped system.

Response to Arbitrary force/General dynamic loading:



Any force P(t) varying arbitrarily with time can be represented as sequence of infinitesimally short impulse.

The response of linear dynamic system for one of these impulses, the one at time τ of magnitude $I = P(\tau)\cdot d\tau$ is given by:-

unit impulse response function.

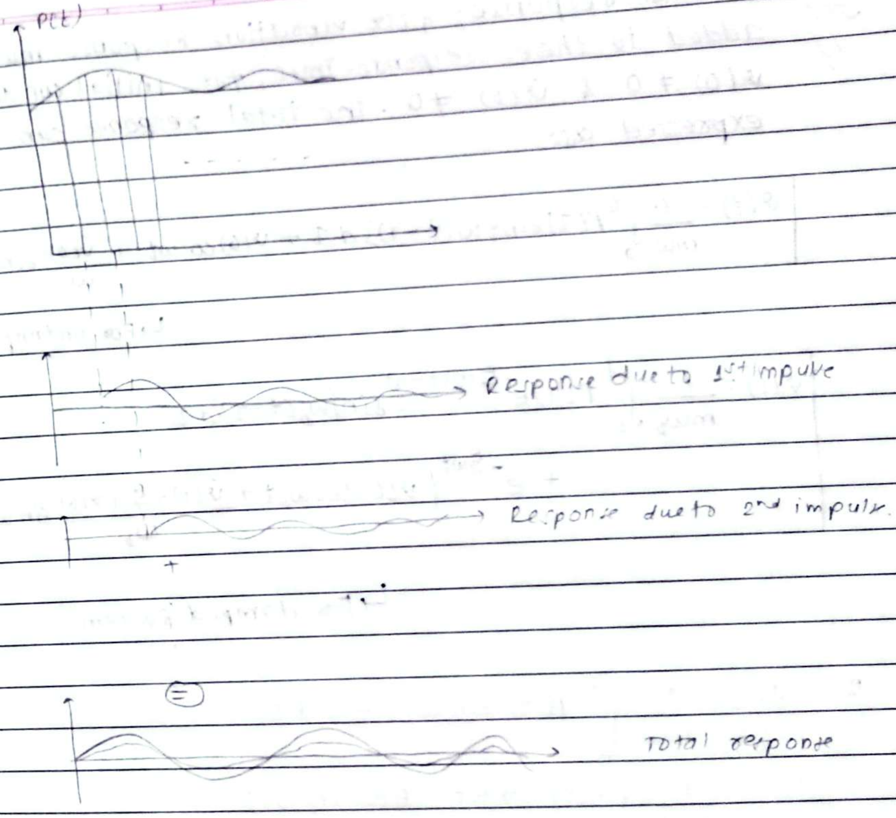
$$dv(t) = I * h(t-\tau)$$

$$\therefore dv(t) = [P(\tau)d\tau] * h(t-\tau) \quad ; \quad t \geq \tau$$

Hence, total response of the system at time 't' is the sum of the responses due to all the impulses upto that time t.

$$\boxed{v(t) = \int_0^t dv(t) = \int_0^t P(\tau)\cdot h(t-\tau)\cdot d\tau}$$

↳ Convolution Integral (Superposition Integral)



→ Now, after substituting for unit impulse response function $h(t-\tau)$ in convolution integral gives
* Duhamel Integral

$$\boxed{v(t) = \frac{1}{m\omega}\int_0^t P(\tau)\sin\{\omega(t-\tau)\}d\tau}$$

↳ Duhamel Integral (for undamped system)

$$\boxed{v(t) = \frac{1}{m\omega_D}\int_0^t P(\tau)\, e^{-\xi\omega(t-\tau)}\cdot\sin\{\omega_D(t-\tau)\}d\tau}$$

↳ Duhamel Integral (for damped system)

For total response; free vibration response must be added to there response. Thus, for initial conditions $v(0) \neq 0$ & $\dot{v}(0) \neq 0$. The total response can be expressed as:

$$\boxed{v(t) = \frac{1}{m\omega}\int_0^t p(\tau)\sin\{\omega(t-\tau)\}\,d\tau + v(0)\cos\omega t + \frac{\dot{v}(0)}{\omega}\sin\omega t}$$

↳ For undamped system.

$$\boxed{v(t) = \frac{1}{m\omega_D}\int_0^t p(\tau)e^{-\xi\omega(t-\tau)}\cdot\sin\{\omega_D(t-\tau)\}\,d\tau + e^{-\xi\omega t}\left[v(0)\cos\omega_D t + \frac{\dot{v}(0)+\xi\omega v(0)}{\omega_D}\sin\omega_D t\right]}$$

↳ For damped system.

$v(t) = \frac{1}{m\omega}\int_0^t p(\tau)\sin\omega(t-\tau)\,d\tau$

$= \frac{1}{m\omega}\left[A(t)\sin\omega t - B(t)\cos\omega t\right]$

where,

$A(t) = \frac{\Delta t}{2}\left(I_0 + 2I_1 + 2I_2 + \cdots + I_n\right)$

$B(t) = \frac{\Delta t}{2}\left(I_0 + 2I_1 + 2I_2 + \cdots + I_n\right)$

$I(t)$ for A, $p(t)\cos(\omega t)$

$I(t)$ for B, $p(t)\sin(\omega t)$

$\Delta t \rightarrow$ assume, (mω)

t	ωt	p(t)	p(t) cos(ωt) $(I(t))\to A$	A(t)	p(t) sin(ωt) $(I(t))\to B$	B(t)	v(t)
0	0	(fig)					(Duhamel contd. at end of page)
Δt							
2Δt							
⋮							

- Discrete fou[rier transfer]
- Fast-fourier transfer (FFT)

Frequency Domain Analysis

↳ Analysis of signals or mathematical function in reference frequency.

$\sin A\cos B+\cos A\sin B = \sin(A+B)$

$\sin A\cos B-\cos A\sin B = \sin(A-B)$

$\cos A\cos B-\sin A\sin B = \cos(A+B)$

$\cos A\cos B+\sin A\sin B = \cos(A-B)$

$\frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1}\cdot f'(x)$

$\frac{d}{dx}[a^{f(x)}] = a^{f(x)}\cdot f'(x)\log_e(a)$

$\frac{d}{dx}[\log x] = \frac{1}{x}$, for $x>0$

$\frac{d}{dx}(\sin x) = \cos x$, $\frac{d}{dx}(\tan x) = \sec^2 x$, $\frac{d}{dx}(\sec x) = \sec x\tan x$

$\frac{d}{dx}(\cos x) = -\sin x$, $\frac{d}{dx}(\text{cosec}\,x) = -\text{cosec}\,x\cot x$, $\frac{d}{dx}(\cot x) = -\text{cose}$

$\frac{d}{dx}\sinh x = \cosh x$; $\frac{d}{dx}\text{sech}\,x = -\text{sech}\,x\tanh x$

$\frac{d}{dx}\cosh x = \sinh x$; $\frac{d}{dx}\text{cosech}\,x = -\text{cosech}\,x\coth x$

$\frac{d}{dx}\tanh x = \text{sech}^2 x$; $\frac{d}{dx}\coth x = -\text{cosech}^2 x$

$\left(\sinh x = \frac{e^x-e^{-x}}{2}\ ;\ \cosh x = \frac{e^x+e^{-x}}{2}\right)$

$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$

$\int \sin x\, dx = -\cos x + c$

$\int \tan x\, dx = \log|\sec x| + c$

$\int \cot x\, dx = \log(\sin x) + c$

$\int \sec x\, dx = \log(\sec x + \tan x) + c$

$\int \text{cosec}\,x\, dx = \log(\text{cosec}\,x - \cot x) + c$

$\int uv\, dx = u\int v\, dx - \int\left(\frac{du}{dx}\int v\, dx\right)dx + c$ (Integration by parts)

$u \rightarrow$ easily derivable, $v \rightarrow$ easily integrable

$u \rightarrow$ ILATE (Inverse, Log, Algebra, trignometric, expon)

U-substitution: $\sqrt{u}$; e^{u}, $\frac{1}{u}$, u^x, $\frac{1}{\sqrt{u}}$, $\frac{1}{u^2}$,

$\log u$, $\sin u$, $\cos u$

$\cos 2A = 2\cos^2 A - 1 = 1 - \sin^2 A = \cos^2 A - \sin^2 A$

$2\sin A\sin B = \cos(A-B) - \cos(A+B)$

$2\cos A\cos B = \cos(A-B) + \cos(A+B)$

$2\sin A\cos B = \sin(A+B) + \sin(A-B)$

$2\cos A\sin B = \sin(A+B) - \sin(A-B)$

Chapter 3.0

Multi Degree of Freedom (MDOF) Systems

MDOF

- local co-ordinate
- Generalised co-ordinate

→ two or more locations of lumped masses to describe their motion → state, response

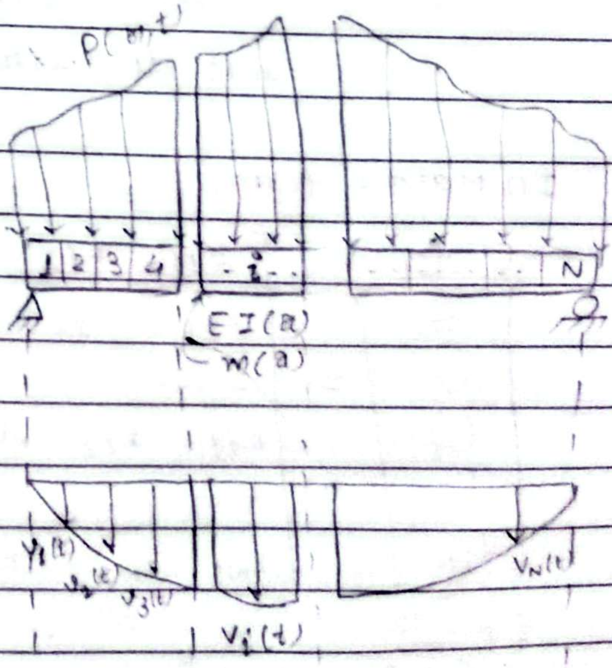
→ All systems in the universe is continuous but with minimal tradeoff in accuracy, if system can be discretized, then it is almost always done so.

→ In general, structural response in general cannot be discribed by SDOF system.

⇒ Continuous system (infinite nodes)

⇒ Lumped-mass system. (Discrete system) (finite nodes/mass points)

*



The system has

→ Force of Inertia, f_{Ii}

→ Force of damping, f_{Di}

→ Force of elasticity, f_{Si}

The equation of dynamic equilibrium will be:-

$$\left.\begin{aligned} f_{I_1} + f_{D_1} + f_{S_1} &= P_1(t) \\ f_{I_2} + f_{D_2} + f_{S_2} &= P_2(t) \\ &\;\vdots \\ f_{IN} + f_{DN} + f_{SN} &= P_N(t) \end{aligned}\right\} \text{---(M1)}$$

In Matrix form/column vector:

$$\{f_I\} + \{f_D\} + \{f_S\} = \{P_t\} \text{ ------ (M2)}$$

where

$$\left.\begin{aligned} f_{S_1} &= k_{11}v_1 + k_{12}v_2 + k_{13}v_3 + \cdots + k_{1N}v_N \\ f_{S_2} &= k_{21}v_1 + k_{22}v_2 + k_{23}v_3 + \cdots + k_{2N}v_N \\ &\;\vdots \\ f_{SN} &= k_{N1}v_1 + k_{N2}v_2 + k_{N3}v_3 + \cdots + k_{NN}v_N \end{aligned}\right] \text{--- (M3)}$$

k_{ij} → stiffness influenced coefficient defined as the force corresponding to node i due to a unit displacement at node j.

In Matrix form,

$$\begin{Bmatrix} f_{S_1} \\ f_{S_2} \\ \vdots \\ f_{S_i} \\ \vdots \\ f_{SN} \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} \cdots & k_{1i} \cdots & k_{1N} \\ k_{21} & k_{22} & k_{23} \cdots & k_{2i} \cdots & k_{2N} \\ \cdots & & & & \cdots \\ k_{i1} & k_{i2} & k_{i3} \cdots & k_{ii} \cdots & k_{iN} \\ \cdots & & & & \cdots \\ k_{N1} & k_{N2} & k_{N3} \cdots & k_{Ni} \cdots & k_{NN} \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \\ \vdots \\ v_i \\ \vdots \\ v_N \end{Bmatrix} \text{ -- (M4)}$$

i.e.

$$\{f_S\} = [K]\{v\} \text{ --- (M5)}$$

$$\left[\begin{aligned} &\text{In SDOF} \\ &f_S = k\,v(t) \\ &f_D = c\,\dot{v}(t) \\ &f_I = m\,\ddot{v}(t) \end{aligned}\right]$$

Similarly,

$$\begin{Bmatrix} f_{D_1} \\ f_{D_2} \\ \vdots \\ f_{D_i} \\ \vdots \\ f_{DN} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \cdots & c_{1i} \cdots & c_{1N} \\ c_{21} & c_{22} & c_{23} \cdots & c_{2i} \cdots & c_{2N} \\ \cdots & & & & \cdots \\ c_{i1} & c_{i2} \cdots & & c_{ii} \cdots & c_{iN} \\ \cdots & & & & \cdots \\ c_{N1} & c_{N2} & \cdots & c_{Ni} \cdots & c_{NN} \end{bmatrix} \begin{Bmatrix} \dot{v}_1 \\ \dot{v}_2 \\ \vdots \\ \dot{v}_i \\ \vdots \\ \dot{v}_N \end{Bmatrix} \text{ --- (M6)}$$

i.e. $\{f_D\} = [C]\{\dot{v}\}$ ---- (M7)

c_{ij} → damping influence coefficient, defined by the force corresponding to coordinate i due to a unit velocity at co-ordinate j.

$$\begin{Bmatrix} f_{I_1} \\ f_{I_2} \\ \vdots \\ f_{Ii} \\ \vdots \\ f_{IN} \end{Bmatrix} = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1i} \cdots & m_{1N} \\ m_{21} & m_{22} & \cdots & m_{2i} \cdots & m_{2N} \\ \cdots & & & & \cdots \\ m_{i1} & m_{i2} & \cdots & m_{ii} \cdots & m_{iN} \\ \cdots & & & & \cdots \\ m_{N1} & m_{N2} & \cdots & m_{Ni} \cdots & m_{NN} \end{bmatrix} \begin{Bmatrix} \ddot{v}_1 \\ \ddot{v}_2 \\ \vdots \\ \ddot{v}_i \\ \vdots \\ \ddot{v}_N \end{Bmatrix} \text{ --- (M8)}$$

i.e. $\{f_I\} = [m]\{\ddot{v}\}$ --- (M9)

m_{ij} → Mass influence coefficient, defined by the force corresponding to co-ordinate i due to a unit acceleration at co-ordinate j.

...ting (M5), (M7) & M(9) in equation (M2)

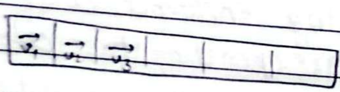
$$[k]\{\bar{v}\} + [c]\{\dot{v}\} + [m]\{\ddot{v}\} = \{P_t\} \quad \text{--- (M10)}$$

which is general equation of motion of MDOF system.

Axial force effect: (or Geometric Effect)

$$\{f_I\} + \{f_D\} + \{f_S\} - \underbrace{\{f_G\}}_{\text{Axial force (geometry force)}} = \{P_t\} \quad \text{--- (M11)}$$

[direction opposite $\{f_I\}$ or $\{f_D\}$ or $\{f_S\}$
i.e. in direction of displacement.



$$\begin{Bmatrix} f_{G1} \\ f_{G2} \\ \vdots \\ f_{Gi} \\ \vdots \\ f_{GN} \end{Bmatrix} = \begin{bmatrix} k_{G11} & k_{G12} & \cdots & k_{G1N} \\ k_{G21} & k_{G22} & \cdots & k_{G2N} \\ \vdots & & & \vdots \\ k_{GN1} & k_{GN2} & \cdots & k_{GNN} \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{Bmatrix} \quad \text{---- (M12)}$$

i.e. $\{f_G\} = [k_G]\{v\}$ --- (M13)

where,

k_{Gij} → Geometric stiffness influence coefficient, defined as the force corresponding to co-ordinate i due to unit displacement of co-ordinate j and resulting from the axial force component in the structure.

Hence, equation (M10) can be revised as:

$$[m]\{\ddot{v}\} + [c]\{\dot{v}\} + [k]\{v\} - [k_G]\{v\} = \{P(t)\} \quad \text{-- (M13)}$$

or,

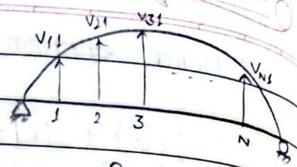
$$[m]\{\ddot{v}\} + [c]\{\dot{v}\} + [\bar{k}]\{v\} = \{P(t)\} \quad \text{---- (M14)}$$

where, $[\bar{k}] = [k] - [k_G]$
↳ combined stiffness matrix.
(may also incorporate shear, torsion, ... effects as well)

* Elastic Properties

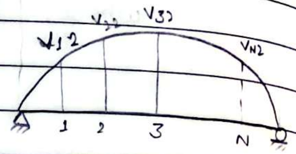
Flexibility:

$V_1 = a_{11}P_1 + a_{12}P_2 + \cdots + a_{1N}P_N$



$P_1 = 1$

$V_2 = a_{21}P_1 + a_{22}P_2 + \cdots + P_{2N}P_N$



$P_2 = 1$

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ \vdots \\ V_N \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{bmatrix} \begin{Bmatrix} P_1 \\ P_2 \\ \vdots \\ \vdots \\ P_N \end{Bmatrix}$$

$\Rightarrow \{V\} = [a]\{P\}.$

Strain Energy,

$U = \frac{1}{2}\sum P_i V_i = \frac{1}{2}\{P\}^T\{V\}$

$\{V\} = [a]\{P\}$

$\therefore U = \frac{1}{2}\{P\}^T[a]\{P\} > 0$

$U^T = U = \left[\frac{1}{2}\{P\}^T\{V\}\right]^T = \frac{1}{2}\{V\}^T\{P\}$

$= \frac{1}{2}\{V\}^T[K]\{V\} > 0$

$[K][a] = [a][K] = [I]$

$[a] = [K]^{-1}$

$[K] = [a]^{-1}$

* Betti's Law:

$\frac{1}{2}\{P\}_a^T\{V\}_b = \frac{1}{2}\{P\}_b^T\{V\}_a$

$\{P\}_a^T[a]\{P\}_b = \{P\}_b^T[a]\{P\}_a$

As both sides of above expression is scalar, we can write,

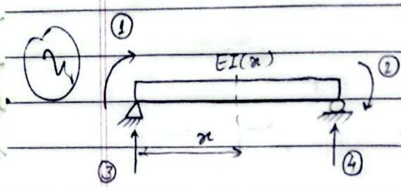
$\{P\}_a^T[a]\{P\}_b = [\{P\}_b^T[a]\{P\}_a]^T$

$\therefore \{P\}_a^T[a]\{P\}_b = \{P\}_a^T[a]\{P\}_b$

$\therefore [a] = [a]^T$ Also, $[K] = [K]^T$

$\therefore a_{ij} = a_{ji}$ i.e. $K_{ij} = K_{ji}$

Interpolation functions (Beam Elements)



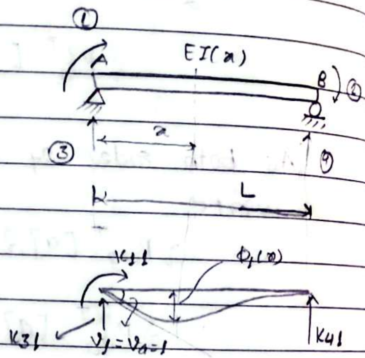
$v(x) = \phi_1(x) v_1 + \phi_2(x) v_2 + \phi_3(x) v_3 + \phi_4(x) v_4$

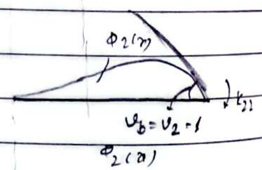
$\phi_1(x) = -x\left(\frac{L-x}{L}\right)^2$

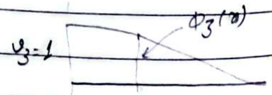
$\phi_2(x) = \frac{-x^2}{L}\left(\frac{x}{L} - 1\right)$

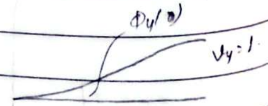
$\phi_3(x) = 1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3$

$\phi_4(x) = 3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3$





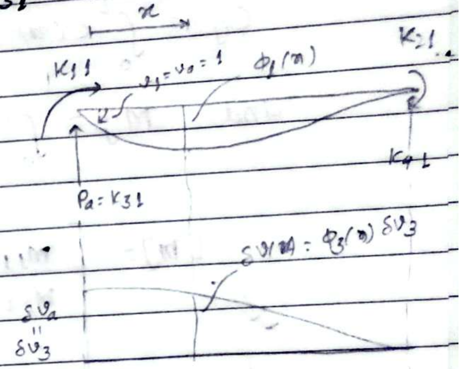




i j

* External work (virtual work)

$W_E = \delta v_a \cdot P_a = \delta v_3 \cdot k_{31}$



* Internal work (virtual)

$$\frac{\partial^2}{\partial x^2}[\delta v(x)] = \phi_3'' \, d v_3$$

$$M = EI(x)\, \phi_1''(x)$$

$$W_I = \int_0^L M(x)\, \phi_3''(x)\, \partial v_3 \cdot dx$$

$$= \delta v_3 \int_0^L EI(x)\, \phi_1''(x)\, \phi_3''(x) \cdot dx$$

Equating external and internal work done,

$$W_E = W_I$$

or, $\delta v_3 \cdot k_{31} = \delta v_3 \int_0^L EI(x)\, \phi_1''(x)\, \phi_3''(x)\, dx$

or, $k_{31} = \int_0^L EI(x)\, \phi_1''(x)\, \phi_3''(x) \cdot dx$

In General,

$$k_{ij} = \int_0^L EI(x)\, \phi_i''(x)\, \phi_j''(x)\, dx$$

and $k_{ij} = k_{ji}$ (symmetrical)

Similarly,

$$C_{ij} = \int_0^L c(x)\,\phi_i(x)\,\phi_j(x)\,dx$$

and $$m_{ij} = \int_0^L m(x)\,\phi_i(x)\,\phi_j(x)\,dx$$

$$[m] = \begin{bmatrix} m_{11} & m_{12} & m_{13} & \cdots & m_{1N} \\ m_{21} & m_{22} & m_{23} & \cdots & m_{2N} \\ \cdots & \cdots & \cdots & & \cdots \\ m_{N1} & m_{N2} & m_{N3} & \cdots & m_{NN} \end{bmatrix}$$

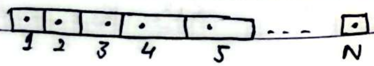
Consistent Mass-Matrix

when $i \neq j$; $m_{ij} \neq 0$

→ For discrete system
(Lumped mass system

m_{11} $m_{21} = 0$ $m_{31} = 0$

$$\begin{cases} m_{ij} \neq 0 \text{ when } i = j \\ m_{ij} = 0 \text{ for } i \neq j \end{cases}$$



$$[m] = \begin{bmatrix} m_{11} & 0 & 0 & \cdots & 0 \\ 0 & m_{22} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & & \cdots \\ 0 & 0 & 0 & \cdots & m_{NN} \end{bmatrix}$$

Lumped mass Matrix.

Eqn of MDOF

$$[m]\{\ddot{v}\} + [c]\{\dot{v}\} + [k]\{v\} = \{P(t)\}$$

$$\boxed{\omega = \sqrt{\frac{k}{m}} \text{ in SDOF}}$$

- Vibration frequencies:-

$$[m]\{\ddot{v}\} + k\{v\} = \{0\} \quad \text{—(i)}$$

→ Undamped free vibration

Solution of above eqn.

$$\{v\} = \{\hat{v}\}\sin(\omega t + \theta) \quad \text{—(ii)}$$

$$\underbrace{\begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix}}_{\text{displacement vector}} = \begin{Bmatrix} \hat{v}_1 \sin(\omega t+\theta) \\ \hat{v}_2 \sin(\omega t+\theta) \\ \hat{v}_3 \sin(\omega t+\theta) \end{Bmatrix} = \underbrace{\begin{Bmatrix} \hat{v}_1 \\ \hat{v}_2 \\ \hat{v}_3 \end{Bmatrix}}_{\text{Amplitude vector (constant)}} \sin(\omega t+\theta)$$

Also,

$$\{\ddot{v}\} = -\omega^2\{\hat{v}\}\sin(\omega t+\theta) \quad \text{—(iii)}$$

Substituting equation (ii) & (iii) in (i),

$$-m\omega^2\{\hat{v}\}\sin(\omega t+\theta) + k\{\hat{v}\}\sin(\omega t+\theta) = \{0\}.$$

or, $$-\omega^2[m]\{\hat{v}\} + [k]\{\hat{v}\} = \{0\}.$$

or, $$([k] - \omega^2[m])\underbrace{\{\hat{v}\}}_{\text{can't be zero.}} = \{0\} \quad \text{—(iv)}$$

The condition for non-trivial solution,

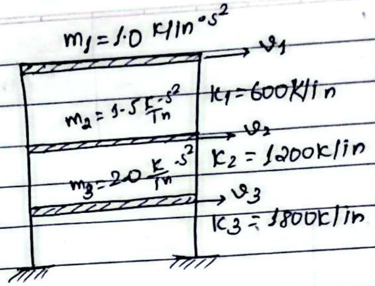
$$\boxed{|[k] - \omega^2 [m]| = 0}$$

This is the frequency eq^n for MDOF system.

$$\{\omega\} = \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_N \end{Bmatrix}$$

The ω's value, we can get depending on DOF.

Eg:

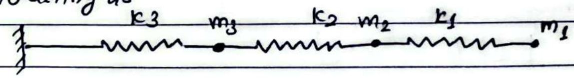


→ Sol^n:

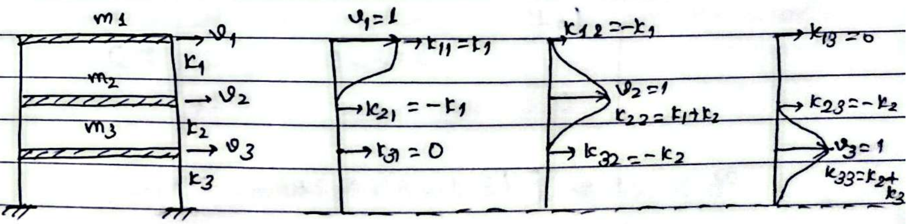
$$[m] = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 2 \end{bmatrix} \frac{k}{in} \cdot s^2$$

$$[k] = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix}$$

Modeling as



'OR'



Thus, $[k] = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 + k_3 \end{bmatrix}$

$$[k] = 600 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 5 \end{bmatrix} k/in.$$

Now,

$$[k] - \omega^2 [m] = 600 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 5 \end{bmatrix} - \omega^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= 600 \begin{bmatrix} 1 - \frac{\omega^2}{600} & -1 & 0 \\ -1 & 3 - \frac{\omega^2}{600} * 1.5 & -2 \\ 0 & -2 & 5 - 2 * \frac{\omega^2}{600} \end{bmatrix}$$

Let, $\frac{\omega^2}{600} = B$

$$\therefore [k] - \omega^2 [m] = 600 \begin{bmatrix} 1 - B & -1 & 0 \\ -1 & 3 - 1.5B & -2 \\ 0 & -2 & 5 - 2B \end{bmatrix}$$

We know,

$$| [k] - \omega^2 [m] | = 0$$

or,

$$\begin{vmatrix} 1 - B & -1 & 0 \\ -1 & 3 - 1.5B & -2 \\ 0 & -2 & 5 - 2B \end{vmatrix} = 0$$

or, $(1-B) \left[(3 - 1.5B)(5 - 2B) - (-2) * (-2) \right]$

$$- (-1) \left[-1 * (5 - 2B) - (-2) * 0 \right] = 0$$

On solving

$B_1 = 0.351$

$B_2 = 1.610$

$B_3 = 3.54$

[values must be arranged in ascending order]

As, $\frac{\omega^2}{600} = B$

$\omega_1^2 = 0.351 * 600 = 210.6$

$\omega_2^2 = 1.610 * 600 = 966$

$\omega_3^2 = 3.54 * 600 = 2124$

$$\Rightarrow \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix} = \begin{Bmatrix} 14.52 \\ 31.08 \\ 46.05 \end{Bmatrix} \text{rad/s.}$$

* Mode shape

From equation (iv)

$$\underbrace{([k] - \omega_n^2 [m])}\{ \hat{v}_n \} = \{0\}$$

$$[E^{(n)}] \{ \hat{v}_n \} = \{0\}$$

where,

$$[E^n] = [k] - \omega_n^2 [m]$$

$$\{ \hat{v}_n \} = \begin{Bmatrix} \hat{v}_{1n} \\ \hat{v}_{2n} \\ \vdots \\ \hat{v}_{Nn} \end{Bmatrix} = \begin{Bmatrix} 1 \\ \hat{v}_{2n} \\ \vdots \\ \hat{v}_{Nn} \end{Bmatrix}$$

Making $\hat{v}_{1n} = 1$

$\boxed{\text{Displacement} = \text{Amplitude} * \text{shape}}$

Matrix in expansion of $[E^n]$

$$\left[\begin{array}{c|cccc} e_{11}^{(n)} & e_{12}^{(n)} & e_{13}^{(n)} & \cdots & e_{1N}^{(n)} \\ \hline e_{21}^{(n)} & e_{22}^{(n)} & e_{23}^{(n)} & \cdots & e_{2N}^{(n)} \\ \vdots & \vdots & \vdots & & \vdots \\ e_{N1}^{(n)} & e_{N2}^{(n)} & e_{N3}^{(n)} & \cdots & e_{NN}^{(n)} \end{array}\right] \begin{Bmatrix} 1 \\ \hline \hat{v}_{2n} \\ \vdots \\ \hat{v}_{Nn} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \hline 0 \\ \vdots \\ 0 \end{Bmatrix}$$

(partitioning given matrix)

$$\left[\begin{array}{c|c} e_{11}^{(n)} & [\tilde{E}_{10}^{(n)}] \\ \hline \{\tilde{E}_{01}^{(n)}\} & [\tilde{E}_{00}^{(n)}] \end{array}\right] \left\{\begin{array}{c} 1 \\ \hline \{\hat{\vartheta}_{0n}\} \end{array}\right\} = \left\{\begin{array}{c} 0 \\ \hline \{0\} \end{array}\right\}$$

column matrix, square matrix, column matrix

Thus,

$$e_{11}^{(n)} + [\tilde{E}_{10}^{(n)}]\{\hat{\vartheta}_{0n}\} = 0 \quad \text{— (a)}$$

$$\{\tilde{E}_{01}^{(n)}\} + [\tilde{E}_{00}^{(n)}]\{\hat{\vartheta}_{0n}\} = \{0\} \quad \text{— (b)}$$

From eq^n (b)

$$\{\hat{\vartheta}_{0n}\} = -[\tilde{E}_{00}^{(n)}]^{-1}\{\tilde{E}_{01}^{(n)}\} \quad \text{— ☆}$$

└→ first $\hat{\vartheta}$ को लाई 1 मानेको हो अनि बाकी remaining shape निकाल्न सकिन्छ (relative to $\hat{\vartheta}_{1n}$)

→ As it gives mode shape we can also represent as

$$\{\phi_n\} = \begin{Bmatrix} \phi_{1n} \\ \phi_{2n} \\ \vdots \\ \phi_{Nn} \end{Bmatrix} = \frac{1}{\hat{\vartheta}_{kn}} \begin{Bmatrix} 1 \\ \hat{\vartheta}_{2n} \\ \vdots \\ \hat{\vartheta}_{Nn} \end{Bmatrix}$$

Mode shape vector for particular mode.

Here, $\hat{\vartheta}_{1n}$.

Overall,

$$[\phi] = [\{\phi_1\} \; \{\phi_2\} \; \{\phi_3\} \; \cdots \; \{\phi_n\}]$$

$$[\phi] = \begin{bmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \vdots & \vdots & & \vdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{bmatrix}$$

Modal matrix or Modeshape Matrix

Problem C^{ntd}

Sol^n, from ☆.

$$\{\phi_{0n}\} = \begin{Bmatrix} \phi_{2n} \\ \phi_{3n} \end{Bmatrix} = -\left(\begin{bmatrix} 3-1.5B & -2 \\ -2 & 5-2B \end{bmatrix}^{-1} \begin{Bmatrix} -1 \\ 0 \end{Bmatrix}\right)$$

For Mode 1,

$$B_1 = \frac{\omega_1^2}{600} = 0.351$$

$$[\tilde{E}_{00}^{(n)}] = \begin{bmatrix} 3-1.5*0.351 & -2 \\ -2 & 5-2*0.351 \end{bmatrix}$$

$$= \begin{bmatrix} 2.475 & -2 \\ -2 & 4.298 \end{bmatrix}$$

$$[\tilde{E}_{00}^{(n)}]^{-1} = \frac{1}{6.638}\begin{bmatrix} 4.298 & 2 \\ 2 & 2.475 \end{bmatrix}$$

Now,

$$\{\phi_1\} = \{\phi_{0n}\} = \begin{Bmatrix} \phi_{2n} \\ \phi_{3n} \end{Bmatrix} = -\frac{1}{6.638}\begin{bmatrix} 4.298 & 2 \\ 2 & 2.475 \end{bmatrix}\begin{Bmatrix} -1 \\ 0 \end{Bmatrix}$$

$$= \begin{Bmatrix} 0.647 \\ 0.301 \end{Bmatrix}$$

$$\therefore \{\phi_1\} = \begin{Bmatrix} \phi_{11} \\ \phi_{21} \\ \phi_{31} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0.647 \\ 0.301 \end{Bmatrix}$$

For Mode 2,

$B_2 = 1.610$

$$\Phi_2 = \begin{Bmatrix} \Phi_{11} \\ \Phi_{21} \end{Bmatrix} - \begin{bmatrix} 3-1.5\times1.610 & -2 \\ -2 & 5-2\times1.610 \end{bmatrix}^{-1} \begin{Bmatrix} -1 \\ 0 \end{Bmatrix}$$

$$\Phi_2 = \begin{Bmatrix} \phi_{12} \\ \phi_{22} \\ \phi_{32} \end{Bmatrix} = \begin{Bmatrix} 1 \\ -0.602 \\ -0.676 \end{Bmatrix}$$

For mode 3

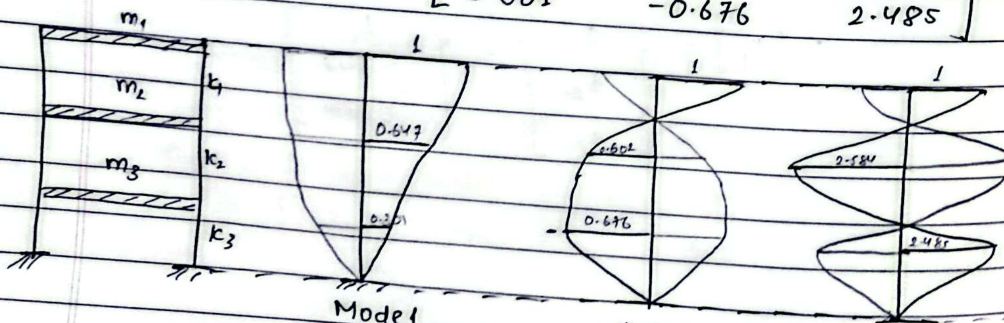
$B_3 = 3.54$

$$\Phi_3 = -\begin{bmatrix} 3-1.5\times3.54 & -2 \\ -2 & 5-2\times3.54 \end{bmatrix}^{-1} \begin{Bmatrix} -1 \\ 0 \end{Bmatrix}$$

$$\Phi_3 = \begin{Bmatrix} \phi_{13} \\ \phi_{23} \\ \phi_{33} \end{Bmatrix} = \begin{Bmatrix} 1 \\ -2.584 \\ 2.485 \end{Bmatrix}.$$

Thus

$$\text{Modal Matrix} = [\Phi] = \begin{bmatrix} 1 & 1 & 1 \\ 0.647 & -0.602 & -2.584 \\ 0.301 & -0.676 & 2.485 \end{bmatrix}$$



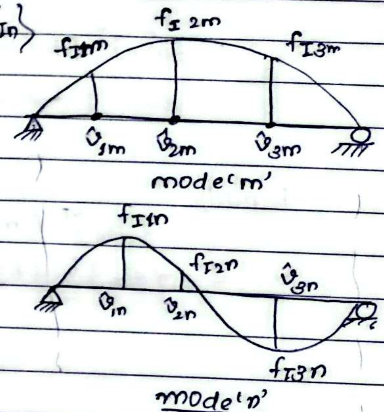
Mode 1 ($\omega_1 = 14.51$ rad/s) Mode 2 ($\omega_2 = 31.08$ rad/s) Mode 3 ($\omega_3 = 46.05$ rad/s)

[vibration m...

Orthogonality Conditions:-

$\{\Phi_1\}, \{\Phi_2\}, \{\Phi_3\} \cdots\cdots \{\Phi_n\}$

$$\underbrace{[k]\{\hat{v}_n\}}_{\text{Elastic resisting force}} = \underbrace{\omega_n^2[m]\{\hat{v}_n\}}_{\text{Inertial force}} = \{f_{In}\}$$



Apply Betti's Law:-

$$-\{f_{Im}\}^T\{\hat{v}_n\} = -\{f_{In}\}^T\{\hat{v}_m\}$$

or, $(\omega_m^2[m]\{\hat{v}_m\})^T\{\hat{v}_n\} = (\omega_n^2[m]\{\hat{v}_n\})^T\{\hat{v}_m\}$

or $\omega_m^2\{\hat{v}_m\}^T[m]^T\{\hat{v}_n\} = \omega_n^2\{\hat{v}_n\}^T[m]^T\{\hat{v}_m\}$

$\because [m]^T = [m]$

$\therefore \omega_m^2\{\hat{v}_m\}^T[m]\{\hat{v}_n\} = \omega_n^2\{\hat{v}_n\}^T[m]\{\hat{v}_m\}$

Transposing right side only as both sides are scalars

or, $\omega_m^2\{\hat{v}_m\}^T[m]\{\hat{v}_n\} = \omega_n^2\{\hat{v}_m\}^T[m]\{\hat{v}_n\}$.

or $(\omega_m^2 - \omega_n^2)\{\hat{v}_m\}^T[m]\{\hat{v}_n\} = 0$

$\because \omega_m \neq \omega_n$

$$\left[\therefore \{\hat{v}_m\}^T[m]\{\hat{v}_n\} = 0\right]$$

↳ Orthogonality ~~cond~~ condition w.r.t. mass.

$$[k]\{\hat{v}_n\} = \omega_n^2 [m]\{\hat{v}_n\}$$

Premultiply both sides by $\{\hat{v}_m\}^T$

$$\{\hat{v}_m\}^T[k]\{\hat{v}_n\} = \omega_n^2 \{\hat{v}_m\}^T[m]\{\hat{v}_n\}$$

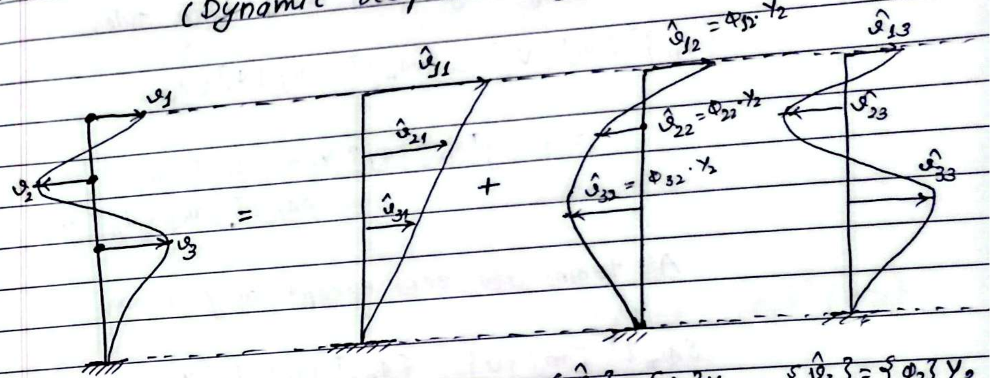
$$\therefore \{\hat{v}_m\}^T[k]\{\hat{v}_n\} = 0$$

↳ 2^{nd} orthogonality condition (w.r.t. stiffness)

Finally,

$$\left.\begin{aligned} \{\phi_m\}^T[m]\{\phi_n\} &= 0 \\ \& \quad \{\phi_m\}^T[k]\{\phi_n\} &= 0 \end{aligned}\right\} \omega_m \neq \omega_n$$

Normal co-ordinates (Dynamic response using superposition)



$\{v\} = [\Phi]\{Y\}$ $\quad \{\hat{v}_1\} = \{\phi_1\}Y_1$ $\quad \{\hat{v}_2\} = \{\phi_2\}Y_2$ $\quad \{\hat{v}_3\} = \{\phi_3\}Y_3$

$$\therefore \{v\} = \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix}$$

$$\{v_1\} = \begin{Bmatrix} \hat{v}_{11} \\ \hat{v}_{21} \\ \hat{v}_{31} \end{Bmatrix} = \begin{Bmatrix} \phi_{11} \\ \phi_{21} \\ \phi_{31} \end{Bmatrix} Y_1$$

Y_n = Normal coordinates (for amplitude of vibration)

Total displacement

$$\{v\} = [\Phi]\{Y\}$$

$$= [\{\phi_1\} \ \{\phi_2\} \ \cdots \ \{\phi_N\}] \begin{Bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{Bmatrix}$$

$$= \{\phi_1\}Y_1 + \{\phi_2\}Y_2 + \cdots + \{\phi_N\}Y_N$$

$$\boxed{\{v\} = \sum_{n=1}^{N} \{\phi_n\} Y_n}$$

$\{v\} = [\Phi]\{Y\}$

Premultiplying by $\{\Phi_n\}^T[m]$ both sides,

$$\{\Phi_n\}^T[m]\{v\} = \{\Phi_n\}^T[m][\Phi]\{Y\}$$

$$= \{\Phi_n\}^T[m]\{\Phi_1\}Y_1 + \{\Phi_n\}^T[m]\{\Phi_2\}Y_2 + \cdots + \{\Phi_n\}^T[m]\{\Phi_n\}Y_N$$

All terms are zero except only term.

$$\{\Phi_n\}^T[m]\{v\} = \{\Phi_n\}^T[m]\{\Phi_n\}Y_n$$

$$\therefore \boxed{Y_n = \frac{\{\Phi_n\}^T[m]\{v\}}{\{\Phi_n\}^T[m]\{\Phi_n\}}}$$

Also,

$$\dot{Y}_n = \frac{\{\Phi_n\}^T[m]\{\dot{v}\}}{\{\Phi_n\}^T[m]\{\Phi_n\}}$$

Uncoupled equation of motion (Undamped case)

$$[m]\{\ddot{v}\} + [k]\{v\} = \{P(t)\}$$

Also, $\{v\} = [\Phi]\{Y\}$

$\{\ddot{v}\} = [\Phi]\{\ddot{Y}\}$

Substitute,

$$[m][\Phi]\{\ddot{Y}\} + [k][\Phi]\{Y\} = \{P(t)\}$$

Premultiply by $\{\Phi_n\}^T$

$$\{\Phi_n\}^T[m][\Phi]\{\ddot{Y}\} + \{\Phi_n\}^T[k][\Phi]\{Y\} = \{\Phi_n\}^T\{P(t)\}$$

$$\Rightarrow \{\Phi_n\}^T[m]\{\Phi_n\}\ddot{Y}_n + \{\Phi_n\}^T[k]\{\Phi_n\}Y = \{\Phi_n\}^T\{P(t)\}$$

or, $\boxed{M_n\ddot{Y}_n + K_nY_n = P_n(t)}$

where,

$M_n = \{\Phi_n\}^T[m]\{\Phi_n\}$

$K_n = \{\Phi_n\}^T[k]\{\Phi_m\}$

$P_n(t) = \{\Phi_n\}^T\{P(t)\}$

Normal co-ordinates for generalised mass, stiffness and load for mode 'n'.

$$[k]\{\hat{v}_n\} = \omega_n^2[m]\{\hat{v}_n\}$$

$$[k]\{\Phi_n\} = \omega_n^2[m]\{\Phi_n\}$$

Premultiply by $\{\Phi_n\}^T$

or, $\{\Phi_n\}^T[k]\{\Phi_n\} = \omega_n^2\{\Phi_n\}^T[m]\{\Phi_n\}$

$$\boxed{K_n = \omega_n^2 M_n}$$

Uncoupled equation of motion (Damped case)

$$[m]\{\ddot{v}\} + [c]\{\dot{v}\} + [k]\{v\} = \{P(t)\}$$

where,

$$\{v\} = [\Phi]\{Y\}$$

$$\therefore \{\dot{v}\} = [\Phi]\{\dot{Y}\}$$

$$\& \{\ddot{v}\} = [\Phi]\{\ddot{Y}\}$$

or $[m][\Phi]\{\ddot{Y}\} + [c][\Phi]\{\dot{Y}\} + [k][\Phi]\{Y\} = \{P(t)\}$

Premultiply by $\{\Phi_n\}^T$

or, $\{\Phi_n\}^T[m][\Phi]\{\ddot{Y}\} + \{\Phi_n\}^T[c][\Phi]\{\dot{Y}\} + \{\Phi_n\}^T[k][\Phi]\{Y\} = \{\Phi_n\}^T\{P(t)\}$

or, $\underbrace{\{\Phi_n\}^T[m]\{\Phi_n\}}_{M_n}\ddot{Y}_n + \underbrace{\{\Phi_n\}^T[c]\{\Phi_n\}}_{C_n}\dot{Y}_n + \underbrace{\{\Phi_n\}^T[k]\{\Phi_n\}}_{K_n}Y_n = \underbrace{\{\Phi_n\}^T\{P(t)\}}_{P_n(t)}$

or, $M_n\ddot{Y}_n + C_n\dot{Y}_n + K_nY_n = P_n(t)$

where,

$$M_n = \{\Phi_n\}^T[m]\{\Phi_n\}$$

$$2\xi_n\omega_n M_n = C_n = \{\Phi_n\}^T[C]\{\Phi_n\}$$

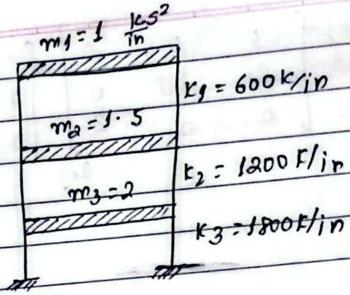
$$\omega_n^2 M_n = K_n = \{\Phi_n\}^T[K]\{\Phi_n\}$$

$$P_n(t) = \{\Phi_n\}^T\{P(t)\}$$

are normal co-ordinates.

or $\ddot{Y}_n + 2\xi_n\omega_n\dot{Y}_n + \omega_n^2 Y_n = \dfrac{P_n(t)}{M_n}$

(Eg)



Calculate $\{Y\}$ & $\{V\}$

Solⁿ:-

$$[m] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$[k] = 600\begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 5 \end{bmatrix} k/in$$

$$\{\omega\} = \begin{Bmatrix} 14.51 \\ 31.08 \\ 46.08 \end{Bmatrix} rad/s, \quad [\Phi] = \begin{bmatrix} 1 & 1 & 1 \\ 0.648 & -0.601 & -2.584 \\ 0.302 & -0.675 & 2.484 \end{bmatrix}$$

Given,

$$\{v\}_{t=0} = \begin{Bmatrix} 0.5 \\ 0.4 \\ 0.3 \end{Bmatrix} in,$$

$$\{\dot{v}\}_{t=0} = \begin{Bmatrix} 0 \\ 9 \\ 0 \end{Bmatrix} in/s$$

Solⁿ:-

$$M_n = \{\Phi_n\}^T[m]\{\Phi_n\}$$

$$\rightarrow M_1 = \{\Phi_1\}^T[m]\{\Phi_1\}$$

$$= \begin{Bmatrix} 0.1 \\ 0.648 \\ 0.302 \end{Bmatrix}^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{Bmatrix} 1 \\ 0.648 \\ 0.302 \end{Bmatrix}$$

$$= \{1 \quad 0.648 \quad 0.308\}_{1\times3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 2 \end{bmatrix}_{3\times3} \begin{Bmatrix} 1 \\ 0.648 \\ 0.302 \end{Bmatrix}_{3\times1}$$

$$= [1.812]$$

$$\rightarrow M_2 = \{\phi_2\}^T [m] \{\phi_2\}$$

$$= \begin{bmatrix} 1 \\ -0.601 \\ -0.675 \end{bmatrix}^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{Bmatrix} 1 \\ -0.601 \\ -0.675 \end{Bmatrix}$$

$$= [2.453]$$

$$\rightarrow M_3 = \{\phi_3\}^T [m] \{\phi_3\}$$

$$= \begin{Bmatrix} 1 \\ -2.584 \\ 2.484 \end{Bmatrix}^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{Bmatrix} 1 \\ -2.584 \\ 2.484 \end{Bmatrix}$$

$$= [23.356]$$
$$= 23.356$$

Now,

$$Y_n(t) = \frac{\{\phi_n\}^T [m] \{v\}}{\{\phi_n\}^T [m] \{\phi_n\}} = \frac{\{\phi_n\}^T [m] \{v\}}{M_n}$$

$$\therefore Y_1(t=0) = \frac{\{\phi_1\}^T [m] \{v\}_{t=0}}{M_1}$$

$$= \frac{\{1 \quad 0.648 \quad 0.302\} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{Bmatrix} 0.5 \\ 0.4 \\ 0.3 \end{Bmatrix}}{1.812}$$

$$= \frac{1.07}{1.812} = 0.59$$

$$Y_2(t=0) = \frac{\{\phi_2\}^T [m] \{v\}_{t=0}}{M_2}$$

$$= \frac{\{1 \quad -0.602 \quad -0.675\} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{Bmatrix} 0.5 \\ 0.4 \\ 0.3 \end{Bmatrix}}{2.453}$$

$$= \frac{-0.2656}{2.453}$$

$$= -0.108$$

$$Y_3(t=0) = \frac{\{\phi_3\}^T [m] \{v\}_{t=0}}{M_3}$$

$$= \frac{\{1 \quad -2.584 \quad 2.484\} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{Bmatrix} 0.5 \\ 0.4 \\ 0.3 \end{Bmatrix}}{23.356}$$

$$= \frac{0.44}{23.356}$$

$$= 0.0188$$

$$\therefore \{Y\}_{t=0} = \begin{Bmatrix} 0.59 \\ -0.108 \\ +0.0188 \end{Bmatrix} \text{ in}$$

Then,

$$\dot{Y}_n(t) = \frac{\{\phi_n\}^T [m] \{\dot{v}\}}{\{\phi_n\}^T [m] \{\phi_n\}} = \frac{\{\phi_n\}^T [m] \{\dot{v}\}}{M_n}$$

$$\dot{Y}_1(t=0) = \frac{\{\phi_1\}^T [m] \{\dot{v}\}}{M_1} = 4.827 \text{ in/s}$$

$$\dot{Y}_2(t=0) = -3.307 \text{ in/s}$$

$$\dot{Y}_3(t=0) = -1.493 \text{ in/s}$$

$$\Rightarrow \{\dot{Y}\}_{t=0} = \begin{Bmatrix} 4.827 \\ -3.307 \\ -1.493 \end{Bmatrix} \text{in/s}$$

Equation of Motion:

$$M_n \ddot{Y}_{n} + K_n Y_n = 0$$

The response will be:

$$Y_n = Y_n(t=0)\cos\omega_n t + \frac{\dot{Y}_n(t=0)}{\omega_n}\sin\omega_n t$$

$$[Y] = [Y_n] = \begin{Bmatrix} Y_1(t) \\ Y_2(t) \\ Y_3(t) \end{Bmatrix} = \begin{Bmatrix} 0.59\cos\omega_1 t \\ -0.108\cos\omega_2 t \\ 0.018\cos\omega_3 t \end{Bmatrix} + \begin{Bmatrix} \frac{4.827}{\omega_1}\sin\omega_1 t \\ \frac{-3.307}{\omega_2}\sin\omega_2 t \\ \frac{-1.493}{\omega_3}\sin\omega_3 t \end{Bmatrix}$$

Then,

For any instant of time,

$$\{v(t)\} = \begin{Bmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \end{Bmatrix} = [\Phi]\{Y\}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0.648 & -0.601 & -2.584 \\ 0.302 & -0.675 & 2.484 \end{bmatrix} \begin{bmatrix} 0.59\cos\omega_1 t + \frac{4.827}{\omega_1}\sin\omega_1 t \\ -0.108\cos\omega_2 t - \frac{3.307}{\omega_2}\sin\omega_2 t \\ 0.018\cos\omega_3 t - \frac{1.493}{\omega_3}\sin\omega_3 t \end{bmatrix}$$

Stodla Method: (Iteration method)

$$([K] - \omega^2[m])\{\hat{v}\} = \{0\}$$

Premultiplying by $\frac{1}{\omega^2}[a]$

or $\left\{\frac{1}{\omega^2}([a][K]) - [a][m]\right\}\{\hat{v}\} = \{0\}$

or $\left(\frac{1}{\omega^2}[I] - [a][m]\right)\{\hat{v}\} = \{0\}$

or $\frac{1}{\omega^2}\{\hat{v}\} = [a][m]\{\hat{v}\}$ — ①

or $\left[\frac{1}{\omega^2}\{\hat{v}\} = [D]\{\hat{v}\}\right]$ — ②

where, $[D] = [a][m]$
$= [K]^{-1}[m]$

called as Dynamic Matrix.

$$\frac{1}{\omega^2}\{v_1^{(1)}\} = [D]\{v_1^{(0)}\}$$

Initial Trial

1st mode

Shape $\{\bar{v}_1^{(1)}\} = [D]\{v_1^{(0)}\}$

where,

$$\frac{1}{\omega^2}\{v_1^{(1)}\} = \{\bar{v}_1^{(1)}\}$$

Displacement

Shape

Find new shape.

Find new shape from initially assumed displacement.

→ $\bar{v}_{k1}^{(s+1)} = \frac{1}{\omega^2} v_{k1}^{(s)}$ ..... (57)

$\therefore \omega^2 = \frac{v_{k1}^{(s)}}{\bar{v}_{k1}^{(s+1)}}$ [In case, $\bar{v}_{k1}^{(s)} = \bar{v}_{k1}^{(s+1)}$]

where, $k_1 = 1, 2, 3, \ldots, n$

After 1st cycle,

$\bar{v}_{k1}^{(1)} = \frac{1}{\omega^2} v_{k1}^{(0)}$

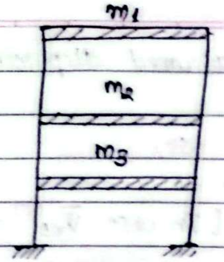
Premultiply by $\{\bar{v}_1^{(1)}\}[m]$

$\{\bar{v}_1^{(1)}\}[m]\{\bar{v}_1^{(1)}\} = \frac{1}{\omega^2}\{\bar{v}_1^{(1)}\}[m]\{v_1^{(0)}\}$

$\Rightarrow \omega^2 = \frac{\{\bar{v}_1^{(1)}\}[m]\{\bar{v}_1^{(0)}\}}{\{\bar{v}_1^{(1)}\}[m]\{\bar{v}_1^{(1)}\}}$

→ $\left(\frac{v_{k1}^{(0)}}{\bar{v}_{k1}^{(1)}}\right)_{min} < \omega^2 < \left(\frac{v_{k1}^{(0)}}{\bar{v}_{k1}^{(1)}}\right)_{max.}$

Example:



$$[m] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 2 \end{bmatrix} \frac{k}{in}\cdot s^2$$

$$[a] = [k^{-1}] = \frac{1}{3600}\begin{bmatrix} 11 & 5 & 2 \\ 5 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix} \frac{in}{kips}$$

$$[D] = [a][m]$$

$$= \frac{1}{3600}\begin{bmatrix} 11 & 5 & 2 \\ 5 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\rightarrow \quad = \underbrace{\frac{1}{3600}\begin{bmatrix} 11 & 7.5 & 4 \\ 5 & 7.5 & 4 \\ 2 & 3 & 4 \end{bmatrix}}_{D} \underbrace{\begin{Bmatrix} 1.0 \\ 1.0 \\ 1.0 \end{Bmatrix}}_{\{v_1^{(0)}\} \rightarrow \text{assumed}}$$

Lets do iteration:

$$\underbrace{\frac{1}{3600}\begin{Bmatrix} 22.5 \\ 16.5 \\ 9.0 \end{Bmatrix} \xrightarrow[\text{(max}^m \text{ लाई 1.0 बनाइदै)}]{\text{Normalizing}} \begin{Bmatrix} 1.0 \\ 0.73 \\ 0.40 \end{Bmatrix}}_{1^{st}\ \text{trial}}$$

$\bar{v}_1^{(1)} \Rightarrow$ shape $\qquad v_1^{(1)}$

→ For 2^{nd} trial

$$D * \{\bar{v}_1^{(1)}\}$$

$$\underbrace{= \frac{1}{3600}\begin{Bmatrix} 18.1 \\ 12.1 \\ 5.8 \end{Bmatrix} \xrightarrow{\text{Normalise}} \begin{Bmatrix} 1.00 \\ 0.67 \\ 0.32 \end{Bmatrix}}_{2^{nd}\ \text{trial}}$$

$\bar{v}_1(2)$ shape $\qquad v_1^{(2)}$

→ 3^{rd} trial

$$D * \{v_1^{(2)}\}$$

$$\underbrace{= \frac{1}{3600}\begin{Bmatrix} 17.31 \\ 11.31 \\ 5.29 \end{Bmatrix} \xrightarrow{\text{Normalise}} \begin{Bmatrix} 1 \\ 0.65 \\ 0.31 \end{Bmatrix}}_{3^{rd}\ \text{trial}}$$

$\bar{v}_1^{(3)} \qquad v_1^{(3)}$

← same

→ 4^{th} trial

$$D * \{v_1^{(3)}\}$$

$$\underbrace{= \frac{1}{3600}\begin{Bmatrix} 17.12 \\ 11.12 \\ 5.19 \end{Bmatrix} \xrightarrow{\text{Normalise}} \begin{Bmatrix} 1 \\ 0.65 \\ 0.31 \end{Bmatrix}}_{4^{th}\ \text{trial}}$$

$\bar{v}_1^{(4)} \qquad v_1^{(4)}$ (Final shape)

→ 5^{th} trial

$$D * \{v_1^{(4)}\}$$

$$= \frac{1}{3600}\begin{Bmatrix} 17.09 \\ 11.09 \\ 5.16 \end{Bmatrix}$$

$$\therefore \omega_1^2 = \frac{v_{11}^{(4)}}{\bar{v}_{11}^{(5)}} = \frac{1.00}{17.09 \times \frac{1}{3600}}$$

$$\omega_1^2 = 210.65$$

i.e. $\omega_1 = 14.51$ rad/s

← Fundamental frequency

After one cycle,

$$(\omega_1)^2_{min} = \frac{v_{11}^{(0)}}{\bar{v}_{11}^{(1)}} = \frac{1}{22.5 \times \frac{1}{3600}} = 160 \text{ rad}^2/\text{sec}^2$$

$$(\omega_1)^2_{max} = \frac{v_{11}^{(0)}}{\bar{v}_{11}^{(1)}} = \frac{1}{9 \times \frac{1}{3600}} = 400 \text{ rad}^2/\text{sec}^2$$

$$\therefore 160 < \omega_1^2 < 400$$

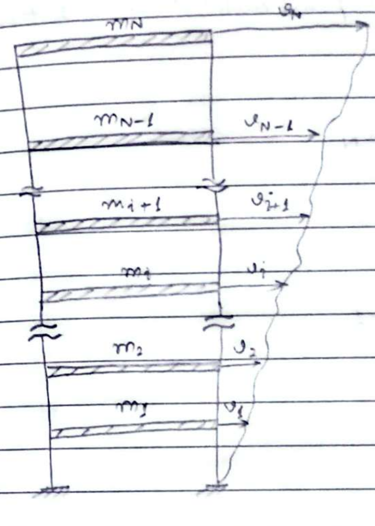
$$12.65 < \omega_1 < 20 \text{ rad/sec.}$$

→ $$\omega_1^2 = \frac{\{\bar{v}_1^{(1)}\}^T [m] \{v_1^{(0)}\}}{\{\bar{v}_1^{(1)}\}^T [m] \{\bar{v}_1^{(1)}\}}$$

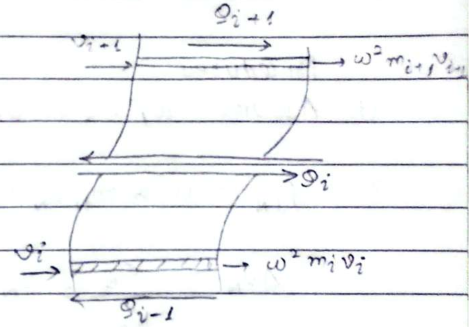
$$= 218.18$$

$$\therefore \omega_1 = 14.77 \text{ rad/sec.}$$

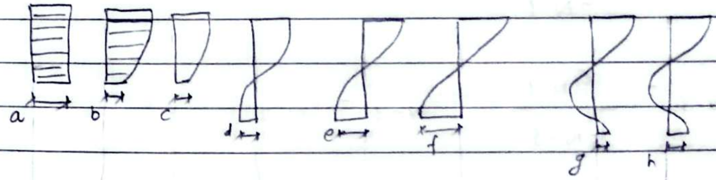
Holzer Method:

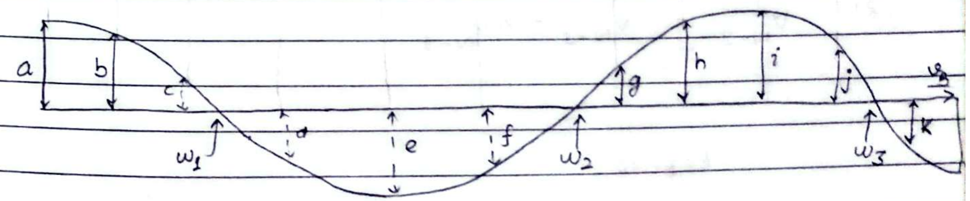


Structural system (fig.1)



Typical storey forces and displacements (fig.2)





Variation of base displacement with applied frequency (maintaining unit displacement at top)

(fig.3)

We assume ω & make displacement as zero to find ω's

Procedures:

(1) Assume ω_1 . & assume ω_N

(2) $f_{IN} = \omega^2 m_N v_N = S_N$

(3) $\Delta v_N = \frac{S_N}{K_N} = v_N - v_{N-1}$

(4) $v_{N-1} = v_N - \Delta v_N$

(5) $f_{I(N-1)} = \omega^2 m_{N-1} v_{N-1}$

(6) $S_{N-1} := \sum_{i=N-1}^{N} f_{Ii}$

(7) $\Delta v_{N-1} = \frac{S_{N-1}}{K_{N-1}} = v_{N-1} - v_{N-2}$

(8) $v_{N-2} = v_{N-1} - \Delta v_{N-1}$

Repeat

(9) $\Delta v_1 = \frac{S_1}{K_1} = v_1 - v_0$

(10) $v_0 = v_1 - \Delta v_1$

if $v_0 = 0 \Rightarrow$ assumed ω is o.k.

Revise ω after 2^{nd} trial as,

$$\left(\frac{\Delta\omega^2}{\Delta v_B}\right)_{1-2} = \left(\frac{\Delta\omega^2}{\Delta v_B}\right)_{2-3}$$

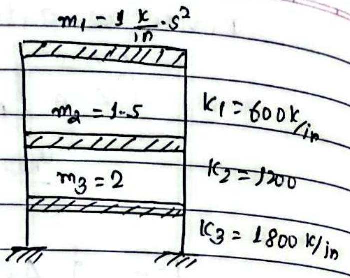
v_{Base}

$$1-2 \begin{cases} \Delta\omega^2 = (\Delta\omega^{(2)})^2 - (\Delta\omega^{(1)})^2 \\ \Delta v_B = v_B^{(1)} - v_B^{(2)} \end{cases}$$

$$2-3 \begin{cases} \Delta\omega^2 = (\Delta\omega^{(3)})^2 - (\Delta\omega^{(2)})^2 \\ \Delta v_B = v_B^{(2)} - 0 \end{cases}$$

(10 तला B building time period → 1 sec)
(3 तला ≈ 0.3-0.4 sec)

Eg:



Trial I

Assume, $\omega^2 = 100\ rad^2/s^2$ (50-100)

	v	Δv	Q	f_I
coordinate 1	1.00			100 (f_{I1})
		0.167 (Q_1/k_1)	100 (Q_1)	
coordinate 2	0.833 ($v_1 - \Delta v_1$)			125 (f_{I2})
		0.187 (Q_2/k_2)	225 (Q_2) ($f_{I1}+f_{I2}$)	
coordinate 3	0.645			129 ($f_{I3} = \omega^2 \times m_3 \times v_3$)
		0.197	354	
Base	0.448 (0.645-0.197)			

Trial 2 ($\omega^2 = 200$)

	v	Δv	Q	f_I
①	1.00			200
		0.333	200	
②	0.667			200.1
		0.333	400.1	
③	0.334			133.6
		0.296	533.7	
base	0.038			

(अन्तिम मान्दा घट्नेछ)
Should be zero at end

After 2nd trial:

$$\left(\frac{\Delta\omega^2}{\Delta v_B}\right)_{1-2} = \left(\frac{\Delta\omega^2}{\Delta v_B}\right)_{2-3}$$

or, $$\frac{200-100}{0.448-0.038} = \frac{\Delta\omega^2}{0.038-0}$$

Thus, $(\Delta\omega^2)_{2-3} = 9.268$

Next,

Trial 3 $\omega^2 = 200 + 9 = 209$

v	Δv	Q	f_I
1.00			209
	0.348	209	
0.652			204.03
	0.344	413.03	
0.308			128.7
	0.301	542	
0.007			

After 3rd trial

$$\left(\frac{\Delta\omega^2}{\Delta v_B}\right)_{2-3} = \left(\frac{\Delta\omega^2}{\Delta v_B}\right)_{3-4}$$

or $$\frac{209-200}{\cancel{0.038} - 0.007} = \frac{(\Delta\omega^2)_{3-4}}{0.007-0}$$

$\Rightarrow \Delta\omega^2 = 2.03$

$\Rightarrow \omega_1^2 = 209 + 2.03 = 211.03$

$\Rightarrow$ $[\omega_1 = 14.52\ rad/s]$ ← fundamental

(ω_2 चाहियो भने ω बढाउँदै जाने s.t v -ve आउँछ)

(Refer for graph fig. 3)

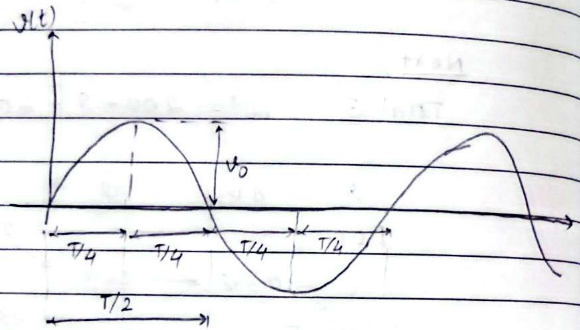
Rayleigh's Method

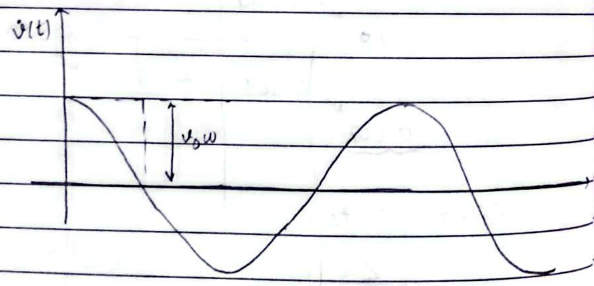
SDOF undamped free vibration

$$\omega = \sqrt{\frac{k}{m}} \quad \text{- - - - - (R1)}$$

$$v = v_0 \sin\omega t \quad \text{- - - - (R 2a)}$$

$$\dot{v} = v_0 \omega \cos\omega t \quad \text{- - - - - (R 2b)}$$





→ Potential Energy, $V = \frac{1}{2} k v^2 = \frac{1}{2} * k * v_0^2 \sin^2\omega t$ — (R3-a)

→ Kinetic Energy, $T = \frac{1}{2} m \dot{v}^2 = \frac{1}{2} m v_0^2 \omega^2 \cos^2\omega t$ -- (R3-b)

→ At, $t = T/_4 = \frac{2\pi}{4\omega} = \frac{\pi}{2\omega}$

$$\longrightarrow V_{max} = \frac{1}{2} k v_0^2 \quad \text{- - - - (R 3-c)}$$

→ At $t = \frac{T}{2} = \frac{2\pi}{2\omega}$.

$$\rightarrow T_{max} = \frac{1}{2} m v_0^2 \omega^2 \quad \text{- - - - (R 3d)}$$

→ Energy conservation principle,

$$V_{max} = T_{max}$$

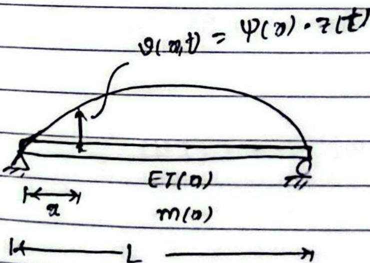
$$\frac{1}{2} k v_0^2 = \frac{1}{2} m v_0^2 \omega^2$$

$$\Rightarrow \left[\omega^2 = \frac{k}{m}\right]$$

$$\Rightarrow \boxed{\omega = \sqrt{\frac{k}{m}}} \rightarrow \text{obtained from Rayleigh's Method}$$

Application → used for approximate frequency analysis for MDOF

Approximate analysis by Rayleigh's Method (for MDOF)



$$v(x,t) = \psi(x) \cdot z(t)$$

$$v(x,t) = \psi(x) \cdot z_0 \sin\omega t \quad \text{—(R5)}$$

$$\dot{v}(x,t) = \psi(x) \cdot z_0 \omega \cos\omega t$$

$$\left\{ \because EI \frac{\partial^2 v}{\partial x^2} = -M \right\}$$

V = Strain energy stored in beam

$$= \frac{1}{2} \int_0^L \frac{M^2 dx}{EI} = \frac{1}{2} \int_0^L \frac{(EI)^2}{EI} \left[\frac{\partial^2 v}{\partial x^2}\right]^2 dx$$

$$= \frac{1}{2} \int_0^L EI(x) \left(\frac{\partial^2 v}{dx^2}\right)^2 \cdot dx \quad \text{- - - (R6)}$$

$\dot{v} = \frac{dv}{dt} = \psi(x) \cdot Z_0 \cdot \omega \cos \omega t$

$\dot{v}_{max} = Z_0 \omega \psi(x)$

$(\dot{v}_{max})^2 = Z_0^2 \omega^2 (\psi(x))^2$

$$\therefore V_{max} = \frac{1}{2} Z_0^2 \int_0^L EI(x) \cdot [\psi''(x)]^2 dx \quad \text{— (R7)}$$

Max^m potential energy

$$T = \frac{1}{2} \int_0^L m(x) \cdot (\dot{v})^2 dx \quad \text{--- R(8)}$$

$$\text{Max}^m \ KE = T_{max} = \frac{1}{2} Z_0^2 \omega^2 \int_0^L m(x) [\psi(x)]^2 dx \quad \text{--- R(9)}$$

Energy conservation principle,

$$V_{max} = T_{max}$$

$$\text{or,} \quad \frac{1}{2} Z_0^2 \int_0^L EI(x) [\psi''(x)]^2 dx = \frac{1}{2} Z_0^2 \omega^2 \int_0^L m(x) (\psi(x))^2 dx$$

$$\therefore \omega^2 = \frac{\int_0^L EI(x) [\psi''(x)]^2 dx}{\int_0^L m(x) [\psi(x)]^2 dx} \quad \text{---- (R10)}$$

* Standard R_{00} method:

$$v^0(x,t) = \psi^{(0)}(x) Z_0^{(1)} \sin \omega t \quad \text{---- (R11)}$$

$$V_{max}^{(0)} = \frac{1}{2} (Z_0^{(0)})^2 \int_0^L EI(x) \left[\psi^{(0)''}(x) \right]^2 dx \quad \text{--- (R12)}$$

$$T_{max}^{(0)} = \frac{1}{2} \int_0^L m(x) \left[\dot{v}^{(0)} \right]^2 dx$$

ref. 8.5 & 8.7 (Clough & Penzin)

$$\therefore T_{max}^{(0)} = \frac{1}{2} (Z_0^{(0)})^2 \omega^2 \int_0^L m(x) [\psi^{(0)}(x)]^2 dx \quad \text{— (R13)}$$

$$\therefore \omega^2 = \frac{\int_0^L EI(x) [\psi^{(0)''}(x)]^2 dx}{\int_0^L m(x) [\psi^{(0)}(x)]^2 dx} \quad \text{--- (R14)}$$

* Standard R_{01} method

The distributed inertia force (at the time of maximum displacement)

$$p^0(x) = \omega^2 m(x) v(0) = \frac{\omega^2 \ \psi(x) \cdot Z_0^{(1)}}{\omega^2} \quad \text{— (R15)}$$

This produce new displacement,

$$v^{(1)} = \omega^2 * \frac{\omega^{(1)}}{\omega^2} = \omega^2 \cdot \frac{\psi(x) \cdot Z_0^{(1)}}{\omega^2}$$

$$= \omega^2 \cdot \psi^{(1)}(x) \cdot \bar{Z}_0^{(1)} \quad \text{---- (R16)}$$

$$v^{(1)}(max) = \frac{1}{2} \int_0^L p^0(x) v^{(1)} \cdot dx$$

$$v^{(1)}(max) = \frac{1}{2} Z_0^{(1)} \cdot \bar{Z}_0^{(1)} \omega^4 \int_0^L m(x) \cdot \psi^{(0)}(x) \cdot \psi'(x) dx \quad \text{— (R17)}$$

$$\underbrace{V_{max}^{(1)}}_{37} = \underbrace{T_{max}^{(0)}}_{13}$$

$$\text{Or,} \quad \frac{1}{2} Z_0^{(0)} \bar{Z}_0^{(1)} \omega^4 \int_0^L (m(x)) \psi^{(0)}(x) \psi'(x) dx = \frac{1}{2} Z_0^{(0)} \cdot \omega^2 \int_0^L * m(x) * [\psi^{(0)} x]^2 dx.$$

$$\Rightarrow \omega^2 = \left. \frac{Z_0^{(0)} \int_0^L m(x) \cdot [\psi^{(0)} x)]^2 dx}{\bar{Z}_0^{(1)} \int_0^L m(x) \cdot \psi^{(0)}(x) \cdot \psi^{(1)}(x) dx} \right] \quad \text{— (R(19)} \quad R_{01} \text{ method}$$

$R_{0\,1}$

kinetic (0), potential (1)

$$T_{max}^{(1)} = \frac{1}{2}\int_0^l m(x)\,[\dot{v}^{(1)}]^2\,dx$$

$$\therefore\ T_{max}^{(1)} = \frac{1}{2}(Z_0^{(1)})^2\cdot\omega^6\int_0^l m(x)\,[\psi'(x)]^2\,dx \quad \text{---(R19)}$$

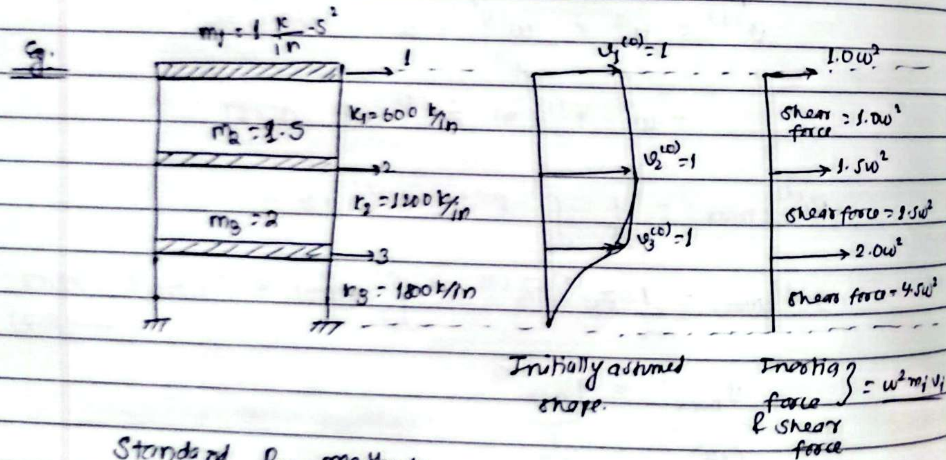
Now,

$$\underset{(R19)}{T_{max}} = \underset{(R17)}{V_{max}^{(1)}}$$

⇒ **R11 Method**

$$\left.\omega^2 = \frac{Z_0^{(0)}\int_0^l m(x)\,\psi^{(0)}(x)\,\psi^{(1)}(x)\,dx}{Z_0^{(1)}\int_0^l m(x)\cdot\psi^{(0)}(x)\cdot\psi^{(1)}(x)\,dx}\right\} \text{—(R20)}$$

Eg.



Standard R00 method:

$$v_1^{(0)} = v_2^{(0)} = v_3^{(0)} = 1 = Z_0^{(0)}\cdot\psi_i^{(0)}$$

$$\text{where, } \psi_i^{(0)} = Z_0^{(0)} = 1.0$$

Max^m K.E. in initial condition:-

$$T_{max}^{(0)} = \frac{1}{2}\sum m_i\,[\dot{v}_i^{(0)}]^2$$

$$= \frac{1}{2}\omega^2\,[Z_0^{(0)}]^2\sum m_i\,[\psi_i^{(0)}]^2$$

ψ → shape. Initial shape $= \begin{Bmatrix}1\\1\\1\end{Bmatrix}$ displacement, assumed $\begin{Bmatrix}1\\1\\1\end{Bmatrix}$

$$= \frac{1}{2}\omega^2\cdot(1)^2\{1\cdot1^2 + 1.5\times1^2 + 2.0\times1^2\}$$

$$= \frac{1}{2}\omega^2\times4.5 = 2.25\,\omega^2.$$

$$V_{max}^{(0)} = \frac{1}{2}\sum k_i\,[\Delta v_i^{(0)}]^2$$

$$= \frac{1}{2}Z_0^{(0)2}\sum k_i\,[\Delta\psi_i^{(0)}]^2$$

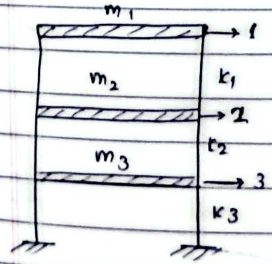
$$= \frac{1}{2}\times1^2\,[600\times0^2 + 1200\times0^2 + 1800\times1^2)$$

$$= 900$$

$$T_{max}^{(0)} = V_{max}^{(0)} \quad \text{(R00 method)}$$

$$\Rightarrow \omega = 20 \text{ rad/s.}$$

Next R01 method.



$$(\omega^2 m_i v_i)\ \Big\}\quad P_i^{(0)} = \omega^2 m_i v_i^{(0)}$$

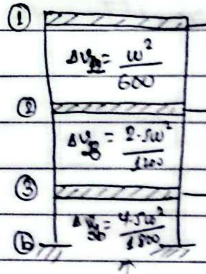
$$P_1^{(0)} = \omega^2 m_1 v_1^{(0)} = \omega^2\times1.0\times1 = \omega^2$$

$$P_2^{(0)} = \omega^2 m_2 v_2^{(0)} = \omega^2\times1.5\times1 = 1.5\omega^2$$

$$P_3^{(0)} = \omega^2 m_3 v_3^{(0)} = \omega^2\times2\times1 = 2\omega^2.$$

$v = (\bar{Z}_0) * \psi$

$\bar{Z}_0 * \omega^2 * \psi$



$v_1^{(1)} = \frac{22.5\omega^2}{3600} = \bar{Z}_0^{(1)}[\psi_1^{(1)}]\omega^2$

$v_2^{(1)} = \frac{16.5\omega^2}{3600} = \bar{Z}_0^{(1)}[\psi_2^{(1)}]\omega^2$

$v_3^{(1)} = \frac{9.0\omega^2}{3600} = \bar{Z}_0^{(1)}[\psi_3^{(1)}] \cdot \omega^2$

SF $\rightarrow \frac{(P_1^{(0)}+P_2^{(0)}+P_3^{(0)})}{1800} = \frac{4.5\omega^2}{1800} = \frac{9\omega^2}{360}$

$v_b = 0$

base

$v_3^{(1)} = \frac{9\omega^2}{3600}$

$v_2^{(1)} = \frac{9\omega^2}{3600} + \frac{2.5\omega^2}{1200} = \frac{16.5\omega^2}{3600}$

$$\left\{ \begin{aligned} \Delta v_{12} &= \frac{\omega^2}{600} = \frac{6\omega^2}{3600} \\ \Delta v_{23} &= \frac{2.5\omega^2}{1200} = \frac{7.5\omega^2}{3600} \\ \Delta v_{3b} &= \frac{4.5\omega^2}{1800} = \frac{9\omega^2}{3600} \end{aligned} \right\}$$

say $v_{base} = 0$

$v_3^{(1)} = v_b + \Delta v_{3b} = \frac{9\omega^2}{3600}$

$v_2^{(1)} = v_3^{(1)} + \Delta v_{23} = \frac{16.5\omega^2}{3600}$

$v_1^{(1)} = v_2^{(1)} + \Delta v_{12} = \frac{16.5\omega^2}{3600} + \frac{6\omega^2}{3600} = \frac{22.5\omega^2}{3600}$

$\therefore V_{max}^{(1)} = \Sigma P_i^{(0)} v_i^{(1)}$

$= \frac{1}{2}\omega^4 \cdot \bar{Z}_0^{(1)} \Sigma(m_i \psi_i^{(0)} \psi_i^{(1)})$

$= \frac{1}{2}\omega^4 \cdot \bar{Z}_0^{(1)} \{1*1*1 + 1.5*1*0.733 + 2*1*0.4\}$

$= \frac{1}{2}\omega^4 \bar{Z}_0^{(1)} * 2.90$

Now, R_{01} method.

$T_{max}^{(0)} = V_{max}^{(1)}$

$\Rightarrow \frac{1}{2}\omega^2 * (4.5) = \frac{1}{2}\omega^4 \cdot \bar{Z}_0^{(1)} (2.90)$

$\omega^2 = \frac{4.5}{2.9} * \frac{1}{\bar{Z}_0^{(1)}}$

$= \frac{4.5}{2.9} * \frac{1}{(22.5/3600)}$

$\omega^2 = 248.27$

$\Rightarrow [\omega = 15.75 \text{ rad/sec}]$

Now, R_{11} method:

$T_{max}^{(1)} = \frac{\omega^6}{2}[\bar{Z}_0^{(1)}] \Sigma m_i (\psi_i^{(1)})^2$

$= \frac{\omega^6}{2}\left[\frac{22.5}{3600}\right]^2 * [1*1^2 + 1.5*0.733^2 + 2*0.4^2]$

$= \frac{\omega^6}{2} * \left(\frac{22.5}{3600}\right)^2 * 2.125$

R_{11} method

$$T_{max}^{(1)} = V_{max}^{(1)}$$

$$\Rightarrow \frac{1}{2} \omega^6 * \left(\frac{22.5}{3600}\right)^2 * (2.125) = \frac{1}{2} \omega^4 * \bar{z}_0^{(1)} (2.90)$$

$$\text{or} \quad \omega^2 = \frac{2.90 * 3600^2}{22.5^2 * 2.125} * \frac{22.5}{3600}$$

$$\text{or} \quad \omega^2 = 218.35$$

$$\Rightarrow \boxed{\omega = 14.77 \text{ rad/sec}}$$